Systems of Differential Equations

Elementary Methods

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Translating a Scalar System to a Vector-Matrix System

Consider the scalar system

$$\begin{aligned} u_1'(t) &= 2u_1(t) + 3u_2(t), \\ u_2'(t) &= 4u_1(t) + 5u_2(t). \end{aligned}$$

Define

$$\vec{u} = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}.$$

Then matrix multiply rules imply that the scalar system is equivalent to the vector-matrix equation

$$\vec{u}' = A\vec{u}$$

Solving a Triangular System

An illustration. Let us solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ for a triangular matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The matrix equation $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ represents two differential equations:

$$\begin{aligned} u_1' &= u_1, \\ u_2' &= 2u_1 + u_2. \end{aligned}$$

The first equation $u_1' = u_1$ has solution $u_1 = c_1 e^t$. The second equation becomes

$$u_2' = 2c_1 e^t + u_2,$$

which is a first order linear differential equation with solution $u_2 = (2c_1 t + c_2)e^t$. The general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ is

$$u_1 = c_1 e^t, \quad u_2 = 2c_1 t e^{-t} + c_2 e^t.$$

Solving a System $\vec{u}' = A\vec{u}$ with Non-Triangular A

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be non-triangular. Then both $b \neq 0$ and $c \neq 0$ must be satisfied. The scalar form of the system $\vec{u}' = A\vec{u}$ is

$$\begin{aligned} u_1' &= au_1 + bu_2, \\ u_2' &= cu_1 + du_2. \end{aligned}$$

Theorem 1 (Solving Non-Triangular $\vec{u}' = A\vec{u}$)
Solutions u_1, u_2 of $\vec{u}' = A\vec{u}$ are linear combinations of the list of Euler solution atoms obtained from the roots r of the quadratic equation

$$\det(A - rI) = 0.$$

Proof of the Non-Triangular Theorem

The method is to differentiate the first equation, then use the equations to eliminate u_2, u_2'. This results in a second order differential equation for u_1. The same differential equation is satisfied also for u_2. The details:

$u_1'' = au_1' + bu_2'$	Differentiate the first equation.
$= au_1' + bcu_1 + bdu_2$	Use equation $u_2' = cu_1 + du_2$.
$= au_1' + bcu_1 + d(u_1' - au_1)$	Use equation $u_1' = au_1 + bu_2$.
$= (a+d)u_1' + (bc - ad)u_1$	Second order equation for u_1 found

The characteristic equation is $r^2 - (a+d)r + (bc - ad) = 0$, which is exactly the expansion of $\det(A - rI) = 0$. The proof is complete.

Cayley-Hamilton-Ziebur Method. The result above extends to any first order homogeneous system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ of differential equations with constant coefficients. The result says that the general solution $\vec{\mathbf{x}}$ is a vector linear combination of the Euler solution atoms found from the roots λ of the characteristic equation $|A - \lambda I| = 0$. Interesting is that the resulting solution $\vec{\mathbf{x}}$ is *real*: no complex numbers appear in the solution $\vec{\mathbf{x}}$.

Shortcut to Solve a Non-Triangular System $\vec{\mathrm{u}}' = A\vec{\mathrm{u}}$

- **Finding u_1.** The two roots r_1, r_2 of the characteristic equation produce two Euler solution atoms,

  In case the roots are distinct, the Euler solution atoms are $e^{r_1 t}$, $e^{r_2 t}$. Then u_1 is a linear combination of atoms: $u_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.

- **Finding u_2.** Isolate u_2 in the first differential equation by division:

$$u_2 = \frac{1}{b}(u_1' - a u_1).$$

The two formulas for u_1, u_2 represent the general solution of the system $\vec{\mathrm{u}}' = A\vec{\mathrm{u}}$, when A is 2×2.

A Non-Triangular Illustration

Let us solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ when A is the non-triangular matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The characteristic polynomial is $\det(A - rI) = (1-r)^2 - 4 = (r+1)(r-3)$. Euler's theorem implies solution atoms e^{-t}, e^{3t}. Then u_1 is a linear combination of the solution atoms, $u_1 = c_1e^{-t} + c_2e^{3t}$.

The first equation $u_1' = u_1 + 2u_2$ implies

$$\begin{aligned} u_2 &= \frac{1}{2}(u_1' - u_1) \\ &= -c_1e^{-t} + c_2e^{3t}. \end{aligned}$$

The general solution of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$ is then

$$u_1 = c_1e^{-t} + c_2e^{3t}, \quad u_2 = -c_1e^{-t} + c_2e^{3t}.$$