Chapter 8

Laplace Transform

Contents

The Laplace transform can be used to solve differential equations. Besides being a different and efficient alternative to variation of parameters and undetermined coefficients, the **Laplace method** is particularly advantageous for input terms that are piecewise-defined, periodic or impulsive.

The **Laplace method** has humble beginnings as an extension of **the method of quadrature** for higher order differential equations and systems. The method is based upon quadrature :

> Multiply the differential equation by the Laplace integrator $dx = e^{-st}dt$ and integrate from $t = 0$ to $t = \infty$. Isolate on the left side of the equal sign the Laplace integral $\int_{t=0}^{t=\infty} y(t)e^{-st}dt$. Look up the answer $y(t)$ in a Laplace integral table.

The **Laplace integral** or the **direct Laplace transform** of a function $f(t)$ defined for $0 \le t < \infty$ is the ordinary calculus integration problem

$$\int_0^\infty f(t)e^{-st}dt,$$

succinctly denoted in science and engineering literature by the symbol

$$\mathcal{L}(f(t)).$$

To the Student. When reading $\mathcal{L}$ in mathematical text, say the words **Laplace of**. Think of the symbol $\mathcal{L}(\cdot)$ as standing for $\int_E (\cdot) dx$ where $(\cdot)$ is the integrand to be substituted, $E = [0, \infty)$ and $dx = e^{-st}dt$ is the **Laplace integrator**. For example, $\mathcal{L}(t^2)$ is shorthand for $\int_0^\infty (t^2)e^{-st}dt$.

The $\mathcal{L}$–notation recognizes that integration always proceeds over $t = 0$ to $t = \infty$ and that the integral involves a fixed *integrator* $e^{-st}dt$ instead of the usual dt. These minor differences distinguish **Laplace integrals** from the ordinary integrals found on the inside covers of calculus texts.

8.1 Introduction to the Laplace Method

The foundation of Laplace theory is **Lerch's cancelation law**

$$(1)\quad \begin{array}{lcl} \int_0^\infty y(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt & \text{implies} & y(t) = f(t), \\ & \text{or} & \\ \mathcal{L}(y(t) = \mathcal{L}(f(t)) & \text{implies} & y(t) = f(t). \end{array}$$

In differential equation applications, $y(t)$ is the unknown appearing in the equation while $f(t)$ is an explicit expression extracted or computed from Laplace integral tables.

An Illustration. Laplace's method will be applied to solve the initial value problem

$$y' = -1, \quad y(0) = 0.$$

No background in Laplace theory is assumed here, only a calculus background is used. The reader should check the answer $y(t) = -t$.

The Plan. The method obtains a relation $\mathcal{L}(y(t)) = \mathcal{L}(-t)$, then Lerch's cancelation law implies that the $\mathcal{L}$-symbols cancel, which gives the differential equation solution $y(t) = -t$.

The **Laplace method** is advertised as a *table lookup method*, in which the solution $y(t)$ to a differential equation is found by looking up the answer in a special integral table. In this sense, the **Laplace method is a generalization of the method of quadrature** to higher order differential equations and systems of differential equations.

Laplace Integral. The integral $\int_0^\infty g(t)e^{-st}dt$ is called the **Laplace integral** of the function $g(t)$. It is defined by $\lim_{N\to\infty} \int_0^N g(t)e^{-st}dt$ and depends on variable s. The ideas will be illustrated for $g(t) = 1$, $g(t) = t$ and $g(t) = t^2$, producing the integral formulas in Table 1, *infra*.

$$\begin{array}{rll} \int_0^\infty (1)e^{-st}dt & = -(1/s)e^{-st}\Big|_{t=0}^{t=\infty} & \text{Laplace integral of } g(t) = 1. \\ & = 1/s & \text{Assumed } s > 0. \\ \int_0^\infty (t)e^{-st}dt & = \int_0^\infty -\frac{d}{ds}(e^{-st})dt & \text{Laplace integral of } g(t) = t. \end{array}$$

$= -\frac{d}{ds}\int_0^\infty (1)e^{-st}dt$	Use $\int \frac{d}{ds}F(t,s)dt = \frac{d}{ds}\int F(t,s)dt$.
$= -\frac{d}{ds}(1/s)$	Use $\mathcal{L}(1) = 1/s$.
$= 1/s^2$	Differentiate.
$\int_0^\infty (t^2)e^{-st}dt = \int_0^\infty -\frac{d}{ds}(te^{-st})dt$	Laplace integral of $g(t) = t^2$.
$= -\frac{d}{ds}\int_0^\infty (t)e^{-st}dt$	
$= -\frac{d}{ds}(1/s^2)$	Use $\mathcal{L}(t) = 1/s^2$.
$= 2/s^3$	

Table 1. The Laplace integral $\int_0^\infty g(t)e^{-st}dt$ for $g(t) = 1$, t and t^2.

$$\int_0^\infty (1)e^{-st}\,dt = \frac{1}{s}, \qquad \int_0^\infty (t)e^{-st}\,dt = \frac{1}{s^2}, \qquad \int_0^\infty (t^2)e^{-st}\,dt = \frac{2}{s^3}.$$

$$\text{In summary,} \quad \mathcal{L}(t^n) = \frac{n!}{s^{1+n}}$$

The Illustration. Details for the **Laplace method** will be given for the solution $y(t) = -t$ of the problem

$$y' = -1, \quad y(0) = 0.$$

Laplace's method (see Table 2) is entirely different from variation of parameters or undetermined coefficients, and it uses only basic calculus and college algebra. In the second Table 3, a succinct version of the first Table 2 is given, using $\mathcal{L}$-notation.

Table 2. Laplace method details for the illustration $y' = -1$, $y(0) = 0$.

$y'(t)e^{-st}dt = -e^{-st}dt$	Multiply $y' = -1$ by $e^{-st}dt$.
$\int_0^\infty y'(t)e^{-st}dt = \int_0^\infty -e^{-st}dt$	Integrate $t = 0$ to $t = \infty$.
$\int_0^\infty y'(t)e^{-st}dt = -1/s$	Use Table 1 forwards.
$s\int_0^\infty y(t)e^{-st}dt - y(0) = -1/s$	Integrate by parts on the left.
$\int_0^\infty y(t)e^{-st}dt = -1/s^2$	Use $y(0) = 0$ and divide.
$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty (-t)e^{-st}dt$	Use Table 1 backwards.
$y(t) = -t$	Apply Lerch's cancelation law. Solution found.

Table 3. Laplace method $\mathcal{L}$-notation details for $y' = -1$, $y(0) = 0$ translated from Table 2.

$\mathcal{L}(y'(t)) = \mathcal{L}(-1)$	Apply $\mathcal{L}$ across $y' = -1$, or multiply $y' = -1$ by $e^{-st}dt$, integrate $t = 0$ to $t = \infty$.
$\mathcal{L}(y'(t)) = -1/s$	Use Table 1 forwards.
$s\mathcal{L}(y(t)) - y(0) = -1/s$	Integrate by parts on the left.
$\mathcal{L}(y(t)) = -1/s^2$	Use $y(0) = 0$ and divide.
$\mathcal{L}(y(t)) = \mathcal{L}(-t)$	Apply Table 1 backwards.
$y(t) = -t$	Invoke Lerch's cancelation law.

In Lerch's law, the formal rule of erasing the integral signs is valid *provided* the integrals are equal for large s and certain conditions hold on y and f — see Theorem 2. The illustration in Table 2 shows that Laplace theory requires an in-depth study of a **special integral table**, a table which is a true extension of the usual table found on the inside covers of calculus books; see Table 1 and section 8.2, Table 4.

The $\mathcal{L}$-notation for the direct Laplace transform produces briefer details, as witnessed by the translation of Table 2 into Table 3. The reader is advised to move from Laplace integral notation to the $\mathcal{L}$–notation as soon as possible, in order to clarify the ideas of the transform method.

Some Transform Rules. The formal properties of calculus integrals plus the integration by parts formula used in Tables 2 and 3 leads to these **rules** for the Laplace transform:

$\mathcal{L}(f(t) + g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$	The integral of a sum is the sum of the integrals.
$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$	Constants c pass through the integral sign.
$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0)$	The t-derivative rule, or integration by parts. See Theorem 3.
$\mathcal{L}(y(t)) = \mathcal{L}(f(t))$ implies $y(t) = f(t)$	Lerch's cancelation law. See Theorem 2.

The first two rules are referenced as **linearity of the transform**. The rules let us manipulate the symbol $\mathcal{L}$ like it was a matrix subject to the rules of matrix algebra. In particular, Laplace's method compares to multiplying a vector equation by a matrix.

Examples

1 **Example (Laplace Method)** Solve by Laplace's method the initial value problem $y' = 5 - 2t$, $y(0) = 1$ to obtain $y(t) = 1 + 5t - t^2$.

Solution: Laplace's method is outlined in Tables 2 and 3. The $\mathcal{L}$-notation of Table 3 will be used to find the solution $y(t) = 1 + 5t - t^2$.

$\mathcal{L}(y'(t)) = \mathcal{L}(5 - 2t)$	Apply $\mathcal{L}$ across $y' = 5 - 2t$.
$= 5\mathcal{L}(1) - 2\mathcal{L}(t)$	Linearity of the transform.
$= \frac{5}{s} - \frac{2}{s^2}$	Use Table 1 forwards.
$s\mathcal{L}(y(t)) - y(0) = \frac{5}{s} - \frac{2}{s^2}$	Apply the parts rule, Theorem 3.
$\mathcal{L}(y(t)) = \frac{1}{s} + \frac{5}{s^2} - \frac{2}{s^3}$	Use $y(0) = 1$ and divide.
$\mathcal{L}(y(t)) = \mathcal{L}(1) + 5\mathcal{L}(t) - \mathcal{L}(t^2)$	Apply Table 1 backwards.
$= \mathcal{L}(1 + 5t - t^2)$	Linearity of the transform.
$y(t) = 1 + 5t - t^2$	Use Lerch's cancelation law.

2 **Example (Laplace Method)** Solve by Laplace's method the initial value problem $y'' = 10$, $y(0) = y'(0) = 0$ to obtain $y(t) = 5t^2$.

Solution: The $\mathcal{L}$-notation of Table 3 will be used to find the solution $y(t) = 5t^2$.

$\mathcal{L}(y''(t)) = \mathcal{L}(10)$	Apply $\mathcal{L}$ across $y'' = 10$.
$s\mathcal{L}(y'(t)) - y'(0) = \mathcal{L}(10)$	Apply the parts rule to y', that is, replace f by y' in Theorem 3.
$s[s\mathcal{L}(y(t)) - y(0)] - y'(0) = \mathcal{L}(10)$	Repeat the parts rule, on y.
$s^2\mathcal{L}(y(t)) = \mathcal{L}(10)$	Use $y(0) = y'(0) = 0$.
$\mathcal{L}(y(t)) = \frac{10}{s^3}$	Use Table 1 forwards. Then divide.
$\mathcal{L}(y(t)) = \mathcal{L}(5t^2)$	Apply Table 1 backwards.
$y(t) = 5t^2$	Invoke Lerch's cancelation law.

Existence of the Transform

The Laplace integral $\int_0^\infty e^{-st} f(t)\,dt$ is known to exist in the sense of the improper integral definition[1]

$$\int_0^\infty g(t)dt = \lim_{N\to\infty} \int_0^N g(t)dt$$

[1]An advanced calculus background is assumed for the Laplace transform existence proof. Applications of Laplace theory require only a calculus background.

provided $f(t)$ belongs to a class of functions known in the literature as functions of **exponential order**. For this class of functions the relation

$$\lim_{t\to\infty} \frac{f(t)}{e^{at}} = 0 \tag{2}$$

is required to hold for some real number a, or equivalently, for some constants M and α,

$$|f(t)| \le Me^{\alpha t}. \tag{3}$$

In addition, $f(t)$ is required to be **piecewise continuous** on each finite subinterval of $0 \le t < \infty$, a term defined as follows.

Definition 1 (Piecewise Continuous)
A function $f(t)$ is **piecewise continuous** on a finite interval $[a,b]$ provided there exists a partition $a = t_0 < \cdots < t_n = b$ of the interval $[a,b]$ and functions f_1, f_2, $\ldots$, f_n continuous on $(-\infty,\infty)$ such that for t not a partition point

$$f(t) = \begin{cases} f_1(t) & t_0 < t < t_1, \\ \vdots & \quad\vdots \\ f_n(t) & t_{n-1} < t < t_n. \end{cases} \tag{4}$$

The values of f at partition points are undecided by equation (4). In particular, equation (4) implies that $f(t)$ has one-sided limits at each point of $a < t < b$ and appropriate one-sided limits at the endpoints. Therefore, f has at worst a **jump discontinuity** at each partition point.

3 Example (Exponential Order) Show that $f(t) = e^t \cos t + t$ is of exponential order.

Solution: The proof must show that $f(t)$ is piecewise continuous and then find an $\alpha > 0$ such that $\lim_{t\to\infty} f(t)/e^{\alpha t} = 0$.

Already, $f(t)$ is continuous, hence piecewise continuous.

From L'Hospital's rule in calculus, $\lim_{t\to\infty} p(t)/e^{\alpha t} = 0$ for any polynomial p and any $\alpha > 0$. Choose $\alpha = 2$, then

$$\lim_{t\to\infty} \frac{f(t)}{e^{2t}} = \lim_{t\to\infty} \frac{\cos t}{e^t} + \lim_{t\to\infty} \frac{t}{e^{2t}} = 0.$$

Theorem 1 (Existence of $\mathcal{L}(f)$)
Let $f(t)$ be piecewise continuous on every finite interval in $t \ge 0$ and satisfy $|f(t)| \le Me^{\alpha t}$ for some constants M and α. Then $\mathcal{L}(f(t))$ exists for $s > \alpha$ and $\lim_{s\to\infty} \mathcal{L}(f(t)) = 0$.

Proof: It has to be shown that the Laplace integral of f is finite for $s > \alpha$. Advanced calculus implies that it is sufficient to show that the integrand is absolutely bounded above by an integrable function $g(t)$. Take $g(t) = Me^{-(s-\alpha)t}$. Then $g(t) \geq 0$. Furthermore, g is integrable, because

$$\int_0^\infty g(t)dt = \frac{M}{s-\alpha}.$$

Inequality $|f(t)| \leq Me^{\alpha t}$ implies the absolute value of the Laplace transform integrand $f(t)e^{-st}$ is estimated by

$$\left|f(t)e^{-st}\right| \leq Me^{\alpha t}e^{-st} = g(t).$$

The limit statement $\lim_{s\to\infty} \mathcal{L}(f(t)) = 0$ follows from $|\mathcal{L}(f(t))| \leq \int_0^\infty g(t)dt = \frac{M}{s-\alpha}$, because the right side of this inequality has limit zero at $s = \infty$. The proof is complete.

Theorem 2 (Lerch)
If $f_1(t)$ and $f_2(t)$ are continuous, of exponential order and $\int_0^\infty f_1(t)e^{-st}dt = \int_0^\infty f_2(t)e^{-st}dt$ for all $s > s_0$, then $f_1(t) = f_2(t)$ for $t \geq 0$.

Proof: See Widder [**?**].

Theorem 3 (Parts Rule or t-Derivative Rule)
If $f(t)$ is continuous, $\lim\limits_{t\to\infty} f(t)e^{-st} = 0$ for all large values of s and $f'(t)$ is piecewise continuous and of exponential order, then $\mathcal{L}(f'(t))$ exists for all large s and $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$.

Proof: See page 615.

Theorem 4 (Euler Atoms have Laplace Integrals)
Let $f(t)$ be the real or imaginary part of $x^n e^{ax+ibx}$, where $b \geq 0$ and a are real and $n \geq 0$ is an integer. Briefly, f is an **Euler Solution Atom**. Then f is of exponential order and $\mathcal{L}(f(t))$ exists. Further, if $g(t)$ is a linear combination of atoms, then $\mathcal{L}(g(t))$ exists.

Proof: By calculus, $\ln|x| \leq 2x$ for $x \geq 1$. Define $c = 2|n| + |a|$. Then $|f(t)| = e^{n\ln|x|+ax} \leq e^{cx}$ for $x \geq 1$, which proves f is of exponential order. The proof is complete.

Remark. Because solutions to undetermined coefficient problems are a linear combination of Euler solution atoms, then Laplace's method applies to all such differential equations. This is the class of all constant-coefficient higher order linear differential equations, and all systems of differential equations with constant coefficients, having a forcing term which is a linear combination of Euler solution atoms.

Exercises 8.1

Laplace method. Solve the given initial value problem using Laplace's method.

1. $y' = -2$, $y(0) = 0$.

2. $y' = 1$, $y(0) = 0$.

3. $y' = -t$, $y(0) = 0$.

4. $y' = t$, $y(0) = 0$.

5. $y' = 1 - t$, $y(0) = 0$.

6. $y' = 1 + t$, $y(0) = 0$.

7. $y' = 3 - 2t$, $y(0) = 0$.

8. $y' = 3 + 2t$, $y(0) = 0$.

9. $y'' = -2$, $y(0) = y'(0) = 0$.

10. $y'' = 1$, $y(0) = y'(0) = 0$.

11. $y'' = 1 - t$, $y(0) = y'(0) = 0$.

12. $y'' = 1 + t$, $y(0) = y'(0) = 0$.

13. $y'' = 3 - 2t$, $y(0) = y'(0) = 0$.

14. $y'' = 3 + 2t$, $y(0) = y'(0) = 0$.

Exponential order. Show that $f(t)$ is of exponential order, by finding a constant $\alpha \ge 0$ in each case such that $\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$.

15. $f(t) = 1 + t$

16. $f(t) = e^t \sin(t)$

17. $f(t) = \sum_{n=0}^{N} c_n x^n$, for any choice of the constants $c_0, \ldots, c_N$.

18. $f(t) = \sum_{n=1}^{N} c_n \sin(nt)$, for any choice of the constants $c_1, \ldots, c_N$.

Existence of transforms. Let $f(t) = te^{t^2}\sin(e^{t^2})$. Establish these results.

19. The function $f(t)$ is not of exponential order.

20. The Laplace integral of $f(t)$, $\int_0^\infty f(t)e^{-st}dt$, converges for all $s > 0$.

Jump Magnitude. For f piecewise continuous, define the **jump** at t by

$$J(t) = \lim_{h\to 0+} f(t+h) - \lim_{h\to 0+} f(t-h).$$

Compute $J(t)$ for the following f.

21. $f(t) = 1$ for $t \ge 0$, else $f(t) = 0$

22. $f(t) = 1$ for $t \ge 1/2$, else $f(t) = 0$

23. $f(t) = t/|t|$ for $t \ne 0$, $f(0) = 0$

24. $f(t) = \sin t/|\sin t|$ for $t \ne n\pi$, $f(n\pi) = (-1)^n$

Taylor series. The series relation $\mathcal{L}(\sum_{n=0}^{\infty} c_n t^n) = \sum_{n=0}^{\infty} c_n \mathcal{L}(t^n)$ often holds, in which case the result $\mathcal{L}(t^n) = n!s^{-1-n}$ can be employed to find a series representation of the Laplace transform. Use this idea on the following to find a series formula for $\mathcal{L}(f(t))$.

25. $f(t) = e^{2t} = \sum_{n=0}^{\infty} (2t)^n/n!$

26. $f(t) = e^{-t} = \sum_{n=0}^{\infty} (-t)^n/n!$

8.2 Laplace Integral Table

The objective in developing a table of Laplace integrals, e.g., Tables 4 and 5, is to keep the table size small. Table manipulation rules appearing in Table 6, page 590, effectively increase the table size manyfold, making it possible to solve typical differential equations from electrical and mechanical models. The combination of Laplace tables plus the table manipulation rules is called the **Laplace transform calculus**.

Table 4 is considered to be a table of minimum size to be memorized. Table 5 adds a number of special-use entries.

Derivations are postponed to page 622. The theory of the generalized factorial function, the **gamma function** $\Gamma(x)$, appears on page 587. The Dirac delta $\delta(t)$ is defined in Section 8.6, page 619.

Table 4. A minimal forward Laplace integral table with $\mathcal{L}$-notation

$\int_0^\infty (t^n)e^{-st}\,dt = \dfrac{n!}{s^{1+n}}$	$\mathcal{L}(t^n) = \dfrac{n!}{s^{1+n}}$
$\int_0^\infty (e^{at})e^{-st}\,dt = \dfrac{1}{s-a}$	$\mathcal{L}(e^{at}) = \dfrac{1}{s-a}$
$\int_0^\infty (\cos bt)e^{-st}\,dt = \dfrac{s}{s^2+b^2}$	$\mathcal{L}(\cos bt) = \dfrac{s}{s^2+b^2}$
$\int_0^\infty (\sin bt)e^{-st}\,dt = \dfrac{b}{s^2+b^2}$	$\mathcal{L}(\sin bt) = \dfrac{b}{s^2+b^2}$

Table 5. Laplace integral table extension

$\mathcal{L}(H(t-a)) = \dfrac{e^{-as}}{s}\ (a \ge 0)$	Heaviside unit step, defined by $H(t) = \begin{cases} 1 & \text{for } t \ge 0, \\ 0 & \text{otherwise.} \end{cases}$
$\mathcal{L}(\delta(t-a)) = e^{-as}$	Dirac delta, $\delta(t) = dH(t)$. Special usage rules apply.
$\mathcal{L}(\textbf{floor}(t/a)) = \dfrac{e^{-as}}{s(1-e^{-as})}$	Staircase function, $\textbf{floor}(x)$ = greatest integer $\le x$.
$\mathcal{L}(\textbf{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$	Square wave, $\textbf{sqw}(x) = (-1)^{\textbf{floor}(x)}$.
$\mathcal{L}(a\,\textbf{trw}(t/a)) = \dfrac{1}{s^2}\tanh(as/2)$	Triangular wave, $\textbf{trw}(x) = \int_0^x \textbf{sqw}(r)dr$.
$\mathcal{L}(t^\alpha) = \dfrac{\Gamma(1+\alpha)}{s^{1+\alpha}}$	Generalized power function, $\Gamma(1+\alpha) = \int_0^\infty e^{-x}x^\alpha dx$.
$\mathcal{L}(t^{-1/2}) = \sqrt{\dfrac{\pi}{s}}$	Because $\Gamma(1/2) = \sqrt{\pi}$.

Table 6. Minimal forward and backward Laplace integral tables

Forward Table	
$f(t)$	$\mathcal{L}(f(t))$
t^n	$\dfrac{n!}{s^{1+n}}$
e^{at}	$\dfrac{1}{s-a}$
$\cos bt$	$\dfrac{s}{s^2+b^2}$
$\sin bt$	$\dfrac{b}{s^2+b^2}$

Backward Table	
$\mathcal{L}(f(t))$	$f(t)$
$\dfrac{1}{s^{1+n}}$	$\dfrac{t^n}{n!}$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{s}{s^2+b^2}$	$\cos bt$
$\dfrac{1}{s^2+b^2}$	$\dfrac{\sin bt}{b}$

First readers are advised to learn Table 6 and back-burner the other tables. To fully understand Table 5 requires many hours of Laplace use.

Examples

4 **Example (Laplace transform)** Let $f(t) = t(t-5) - \sin 2t + e^{3t}$. Compute $\mathcal{L}(f(t))$ using the forward Laplace table and transform linearity properties.

Solution:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \mathcal{L}(t^2 - 5t - \sin 2t + e^{3t}) && \text{Expand } t(t-5). \\
&= \mathcal{L}(t^2) - 5\mathcal{L}(t) - \mathcal{L}(\sin 2t) + \mathcal{L}(e^{3t}) && \text{Linearity applied.} \\
&= \frac{2}{s^3} - \frac{5}{s^2} - \frac{2}{s^2+4} + \frac{1}{s-3} && \text{Forward Table.}
\end{aligned}$$

5 **Example (Inverse Laplace transform)** Use the backward Laplace table plus transform linearity properties to solve for $f(t)$ in the equation

$$\mathcal{L}(f(t)) = \frac{s}{s^2+16} + \frac{2}{s-3} + \frac{s+1}{s^3}.$$

Solution:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \frac{s}{s^2+16} + 2\frac{1}{s-3} + \frac{1}{s^2} + \frac{1}{2}\frac{2}{s^3} && \text{Convert to table entries.} \\
&= \mathcal{L}(\cos 4t) + 2\mathcal{L}(e^{3t}) + \mathcal{L}(t) + \tfrac{1}{2}\mathcal{L}(t^2) && \text{Backward Laplace table.} \\
&= \mathcal{L}(\cos 4t + 2e^{3t} + t + \tfrac{1}{2}t^2) && \text{Linearity applied.} \\
f(t) &= \cos 4t + 2e^{3t} + t + \tfrac{1}{2}t^2 && \text{Lerch's cancelation law.}
\end{aligned}$$

6 **Example (Heaviside)** Find the Laplace transform of $f(t)$ in Figure 1.

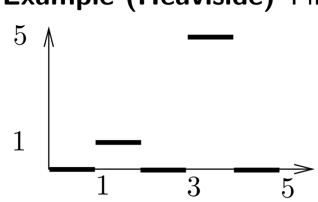

Figure 1. A piecewise defined function $f(t)$ on $0 \le t < \infty$: $f(t) = 0$ except for $1 \le t < 2$ and $3 \le t < 4$.

Solution: The details require the use of the Heaviside function formula

$$H(t-a)-H(t-b)=\begin{cases}1 & a\le t<b,\\ 0 & \text{otherwise.}\end{cases} \tag{1}$$

The formula for $f(t)$:

$$\begin{aligned} f(t) &= \begin{cases}1 & 1\le t<2,\\ 5 & 3\le t<4,\\ 0 & \text{otherwise}\end{cases}\\ &= \begin{cases}1 & 1\le t<2,\\ 0 & \text{otherwise}\end{cases} + 5\begin{cases}1 & 3\le t<4,\\ 0 & \text{otherwise}\end{cases}\\ &= f_1(t)+5f_2(t), \end{aligned}$$

where relation (1) implies

$$f_1(t)=H(t-1)-H(t-2), \quad f_2(t)=H(t-3)-H(t-4).$$

The extended Laplace table gives

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(f_1(t))+5\mathcal{L}(f_2(t)) && \text{Linearity.}\\ &= \mathcal{L}(H(t-1))-\mathcal{L}(H(t-2))+5\mathcal{L}(f_2(t)) && \text{Substitute for } f_1.\\ &= \frac{e^{-s}-e^{-2s}}{s}+5\mathcal{L}(f_2(t)) && \text{Extended table used.}\\ &= \frac{e^{-s}-e^{-2s}+5e^{-3s}-5e^{-4s}}{s} && \text{Similarly for } f_2. \end{aligned}$$

7 Example (Dirac Impulse) A machine shop tool that repeatedly hammers a die is modeled by a Dirac impulse model $f(t)=\sum_{n=1}^{N}\delta(t-n)$. Verify the formula $\mathcal{L}(f(t))=\sum_{n=1}^{N}e^{-ns}$.

Solution:

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}\left(\sum_{n=1}^{N}\delta(t-n)\right)\\ &= \sum_{n=1}^{N}\mathcal{L}(\delta(t-n)) && \text{Linearity.}\\ &= \sum_{n=1}^{N}e^{-ns} && \text{Extended Laplace table.} \end{aligned}$$

8 Example (Square wave) A periodic camshaft force $f(t)$ applied to a mechanical system has the idealized graph shown in Figure 2. Verify formulas $f(t)=1+\textbf{sqw}(t)$ and $\mathcal{L}(f(t))=\frac{1}{s}(1+\tanh(s/2))$.

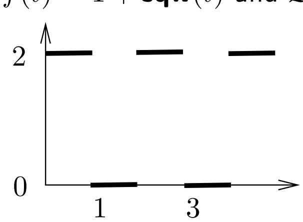

Figure 2. A periodic force $f(t)$ applied to a mechanical system.

Solution:

$$
\begin{aligned}
1+\textbf{sqw}(t) &= \begin{cases} 1+1 & 2n \le t < 2n+1,\ \ n=0,1,\ldots, \\ 1-1 & 2n+1 \le t < 2n+2,\ \ n=0,1,\ldots, \end{cases} \\
&= \begin{cases} 2 & 2n \le t < 2n+1,\ \ n=0,1,\ldots, \\ 0 & \text{otherwise}, \end{cases} \\
&= f(t).
\end{aligned}
$$

By the extended Laplace table, $\mathcal{L}(f(t)) = \mathcal{L}(1) + \mathcal{L}(\textbf{sqw}(t)) = \dfrac{1}{s} + \dfrac{\tanh(s/2)}{s}$.

9 Example (Sawtooth wave) Express the P-periodic sawtooth wave represented in Figure 3 as $f(t) = ct/P - c\,\textbf{floor}(t/P)$ and obtain the formula

$$\mathcal{L}(f(t)) = \frac{c}{Ps^2} - \frac{ce^{-Ps}}{s - se^{-Ps}}.$$

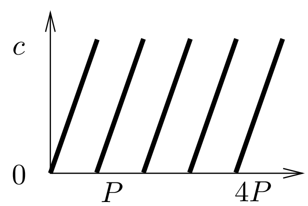

Figure 3. A P-periodic sawtooth wave $f(t)$ of height $c > 0$.

Solution: The representation originates from geometry, because the periodic function f can be viewed as derived from ct/P by subtracting the correct constant from each of intervals $[P, 2P]$, $[2P, 3P]$, etc.

The technique used to verify the identity is to define $g(t) = ct/P - c\,\textbf{floor}(t/P)$ and then show that g is P-periodic and $f(t) = g(t)$ on $0 \le t < P$. Two P-periodic functions equal on the base interval $0 \le t < P$ have to be identical, hence the representation follows.

The fine details: for $0 \le t < P$, $\textbf{floor}(t/P) = 0$ and $\textbf{floor}(t/P + k) = k$. Hence $g(t + kP) = ct/P + ck - c\,\textbf{floor}(k) = ct/P = g(t)$, which implies that g is P-periodic and $g(t) = f(t)$ for $0 \le t < P$.

$$
\begin{aligned}
\mathcal{L}(f(t)) &= \frac{c}{P}\mathcal{L}(t) - c\mathcal{L}(\textbf{floor}(t/P)) && \text{Linearity.} \\
&= \frac{c}{Ps^2} - \frac{ce^{-Ps}}{s - se^{-Ps}} && \text{Basic and extended table applied.}
\end{aligned}
$$

10 Example (Triangular wave) Express the triangular wave f of Figure 4 in terms of the square wave **sqw** and obtain $\mathcal{L}(f(t)) = \dfrac{5}{\pi s^2}\tanh(\pi s/2)$.

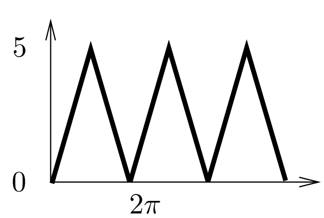

Figure 4. A 2π-periodic triangular wave $f(t)$ of height 5.

Solution: The representation of f in terms of **sqw** is $f(t) = 5\int_0^{t/\pi} \mathbf{sqw}(x)dx$.

Details: A 2-periodic triangular wave of height 1 is obtained by integrating the square wave of period 2. A wave of height c and period 2 is given by $c\,\mathbf{trw}(t) = c\int_0^t \mathbf{sqw}(x)dx$. Then $f(t) = c\,\mathbf{trw}(2t/P) = c\int_0^{2t/P} \mathbf{sqw}(x)dx$ where $c = 5$ and $P = 2\pi$.

Laplace transform details: Use the extended Laplace table as follows.

$$\mathcal{L}(f(t)) = \frac{5}{\pi}\mathcal{L}(\pi\,\mathbf{trw}(t/\pi)) = \frac{5}{\pi s^2}\tanh(\pi s/2).$$

Gamma Function

In mathematical physics, the **Gamma function** or the **generalized factorial function** is given by the identity

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt, \quad x > 0. \tag{2}$$

This function is tabulated and available in computer languages like Fortran, C, C++ and C#. It is also available in computer algebra systems and numerical laboratories, such as `maple, matlab`.

Fundamental Properties of $\Gamma(x)$

The fundamental properties of the generalized factorial function $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$ are as follows.

$$\Gamma(1) = 1 \tag{3}$$

$$\Gamma(1+x) = x\Gamma(x) \tag{4}$$

$$\Gamma(1+n) = n! \quad \text{for integers} \quad n \geq 1. \tag{5}$$

Details for relations (4) and (5): Start with $\int_0^\infty e^{-t}dt = 1$, which gives $\Gamma(1) = 1$. Use this identity and successively relation (4) to obtain relation (5). To prove identity (4), integration by parts is applied, as follows:

$\Gamma(1+x) = \int_0^\infty e^{-t}t^x dt$	Definition.
$= -t^x e^{-t}\big\|_{t=0}^{t=\infty} + \int_0^\infty e^{-t}xt^{x-1}dt$	Use $u = t^x$, $dv = e^{-t}dt$.
$= x\int_0^\infty e^{-t}t^{x-1}dt$	Boundary terms are zero for $x > 0$.
$= x\Gamma(x)$.	

Exercises 8.2

Laplace Transform. Using the basic Laplace table and linearity properties of the transform, compute $\mathcal{L}(f(t))$. Do not use the direct Laplace transform!

1. $\mathcal{L}(2t)$

2. $\mathcal{L}(4t)$

3. $\mathcal{L}(1+2t+t^2)$

4. $\mathcal{L}(t^2-3t+10)$

5. $\mathcal{L}(\sin 2t)$

6. $\mathcal{L}(\cos 2t)$

7. $\mathcal{L}(e^{2t})$

8. $\mathcal{L}(e^{-2t})$

9. $\mathcal{L}(t+\sin 2t)$

10. $\mathcal{L}(t-\cos 2t)$

11. $\mathcal{L}(t+e^{2t})$

12. $\mathcal{L}(t-3e^{-2t})$

13. $\mathcal{L}((t+1)^2)$

14. $\mathcal{L}((t+2)^2)$

15. $\mathcal{L}(t(t+1))$

16. $\mathcal{L}((t+1)(t+2))$

17. $\mathcal{L}(\sum_{n=0}^{10} t^n/n!)$

18. $\mathcal{L}(\sum_{n=0}^{10} t^{n+1}/n!)$

19. $\mathcal{L}(\sum_{n=1}^{10} \sin nt)$

20. $\mathcal{L}(\sum_{n=0}^{10} \cos nt)$

Inverse Laplace transform. Solve the given equation for the function $f(t)$. Use the basic table and linearity properties of the Laplace transform.

21. $\mathcal{L}(f(t)) = s^{-2}$

22. $\mathcal{L}(f(t)) = 4s^{-2}$

23. $\mathcal{L}(f(t)) = 1/s + 2/s^2 + 3/s^3$

24. $\mathcal{L}(f(t)) = 1/s^3 + 1/s$

25. $\mathcal{L}(f(t)) = 2/(s^2+4)$

26. $\mathcal{L}(f(t)) = s/(s^2+4)$

27. $\mathcal{L}(f(t)) = 1/(s-3)$

28. $\mathcal{L}(f(t)) = 1/(s+3)$

29. $\mathcal{L}(f(t)) = 1/s + s/(s^2+4)$

30. $\mathcal{L}(f(t)) = 2/s - 2/(s^2+4)$

31. $\mathcal{L}(f(t)) = 1/s + 1/(s-3)$

32. $\mathcal{L}(f(t)) = 1/s - 3/(s-2)$

33. $\mathcal{L}(f(t)) = (2+s)^2/s^3$

34. $\mathcal{L}(f(t)) = (s+1)/s^2$

35. $\mathcal{L}(f(t)) = s(1/s^2 + 2/s^3)$

36. $\mathcal{L}(f(t)) = (s+1)(s-1)/s^3$

37. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{1+n}$

38. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} n!/s^{2+n}$

39. $\mathcal{L}(f(t)) = \sum_{n=1}^{10} \frac{n}{s^2+n^2}$

40. $\mathcal{L}(f(t)) = \sum_{n=0}^{10} \frac{s}{s^2+n^2}$

Laplace Table Extension. Compute the indicated Laplace integral using the extended Laplace table, page 583.

41. $\mathcal{L}(H(t-2)+2H(t))$

42. $\mathcal{L}(H(t-3)+4H(t))$

43. $\mathcal{L}(H(t-\pi)(H(t)+H(t-1)))$

44. $\mathcal{L}(H(t-2\pi)+3H(t-1)H(t-2))$

45. $\mathcal{L}(\delta(t-2))$

46. $\mathcal{L}(5\delta(t-\pi))$

47. $\mathcal{L}(\delta(t-1)+2\delta(t-2))$

48. $\mathcal{L}(\delta(t-2)(5+H(t-1)))$

49. $\mathcal{L}(\textbf{floor}(3t))$

50. $\mathcal{L}(\textbf{floor}(2t))$

51. $\mathcal{L}(5\,\textbf{sqw}(3t))$

52. $\mathcal{L}(3\,\textbf{sqw}(t/4))$

53. $\mathcal{L}(4\,\textbf{trw}(2t))$

54. $\mathcal{L}(5\,\textbf{trw}(t/2))$

55. $\mathcal{L}(t + t^{-3/2} + t^{-1/2})$

56. $\mathcal{L}(t^3 + t^{-3/2} + 2t^{-1/2})$

Inverse Laplace, Extended Table. Compute $f(t)$, using the extended Laplace integral table.

57. $\mathcal{L}(f(t)) = e^{-s}/s$

58. $\mathcal{L}(f(t)) = 5e^{-2s}/s$

59. $\mathcal{L}(f(t)) = e^{-2s}$

60. $\mathcal{L}(f(t)) = 5e^{-3s}$

61. $\mathcal{L}(f(t)) = \dfrac{e^{-s/3}}{s(1-e^{-s/3})}$

62. $\mathcal{L}(f(t)) = \dfrac{e{-}2s}{s(1-e^{-2s})}$

63. $\mathcal{L}(f(t)) = \dfrac{4\tanh(s)}{s}$

64. $\mathcal{L}(f(t)) = \dfrac{5\tanh(3s)}{2s}$

65. $\mathcal{L}(f(t)) = \dfrac{4\tanh(s)}{3s^2}$

66. $\mathcal{L}(f(t)) = \dfrac{5\tanh(2s)}{11s^2}$

67. $\mathcal{L}(f(t)) = \dfrac{1}{\sqrt{s}}$

68. $\mathcal{L}(f(t)) = \dfrac{1}{\sqrt{s^3}}$

8.3 Laplace Transform Rules

In Table 7, the basic table manipulation rules are summarized. Full statements and proofs of the rules appear in section 8.5, page 613.

The rules are applied here to several key examples. Partial fraction expansions do not appear here, but in section 8.4, in connection with Heaviside's coverup method.

Table 7. Laplace transform rules

$\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$	Linearity. The Laplace of a sum is the sum of the Laplaces.
$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t))$	Linearity. Constants move through the $\mathcal{L}$-symbol.
$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0)$	The t-derivative rule. Derivatives $\mathcal{L}(y')$ are replaced in transformed equations.
$\mathcal{L}\left(\int_0^t g(x)dx\right) = \frac{1}{s}\mathcal{L}(g(t))$	The t-integral rule.
$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t))$	The s-differentiation rule. Multiplying f by t applies $-d/ds$ to the transform of f.
$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))\vert_{s\to(s-a)}$	First shifting rule. Multiplying f by e^{at} replaces s by $s-a$.
$\mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f(t))$, $\mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a))$	Second shifting rule. First and second forms.
$\mathcal{L}(f(t)) = \frac{\int_0^P f(t)e^{-st}dt}{1-e^{-Ps}}$	Rule for P-periodic functions. Assumed here is $f(t+P)=f(t)$.
$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}((f*g)(t))$	Convolution rule. Define $(f*g)(t) = \int_0^t f(x)g(t-x)dx$.

Examples

11 Example (Harmonic oscillator) Solve the initial value problem $x''+x=0$, $x(0)=0$, $x'(0)=1$ by Laplace's method.

Solution: The solution is $x(t)=\sin t$. The details:

$\mathcal{L}(x'') + \mathcal{L}(x) = \mathcal{L}(0)$	Apply $\mathcal{L}$ across the equation.
$s\mathcal{L}(x') - x'(0) + \mathcal{L}(x) = 0$	Use the t-derivative rule.
$s[s\mathcal{L}(x) - x(0)] - x'(0) + \mathcal{L}(x) = 0$	Use again the t-derivative rule.
$(s^2+1)\mathcal{L}(x) = 1$	Use $x(0)=0$, $x'(0)=1$.
$\mathcal{L}(x) = \frac{1}{s^2+1}$	Divide.
$= \mathcal{L}(\sin t)$	Basic Laplace table.
$x(t) = \sin t$	Invoke Lerch's cancellation law.

12 Example (s-differentiation rule) Show the steps for $\mathcal{L}(t^2\, e^{5t}) = \dfrac{2}{(s-5)^3}$.

Solution:

$\mathcal{L}(t^2 e^{5t}) = \left(-\dfrac{d}{ds}\right)\left(-\dfrac{d}{ds}\right)\mathcal{L}(e^{5t})$	Apply s-differentiation.
$= (-1)^2 \dfrac{d}{ds}\dfrac{d}{ds}\left(\dfrac{1}{s-5}\right)$	Basic Laplace table.
$= \dfrac{d}{ds}\left(\dfrac{-1}{(s-5)^2}\right)$	Calculus power rule.
$= \dfrac{2}{(s-5)^3}$	Identity verified.

13 Example (First shifting rule) Show the steps for $\mathcal{L}(t^2\, e^{-3t}) = \dfrac{2}{(s+3)^3}$.

Solution:

$\mathcal{L}(t^2 e^{-3t}) = \mathcal{L}(t^2)\big\|_{s\to s-(-3)}$	First shifting rule.
$= \left(\dfrac{2}{s^{2+1}}\right)\Big\|_{s\to s-(-3)}$	Basic Laplace table.
$= \dfrac{2}{(s+3)^3}$	Identity verified.

14 Example (Second shifting rule) Show the steps for

$$\mathcal{L}(\sin t\, H(t-\pi)) = \frac{-e^{-\pi s}}{s^2+1}.$$

Solution: The second shifting rule is applied as follows.

LHS $= \mathcal{L}(\sin t\, H(t-\pi))$	Left side of the identity.
$= \mathcal{L}(g(t)H(t-a))$	Choose $g(t) = \sin t$, $a = \pi$.
$= e^{-as}\mathcal{L}(g(t+a)$	Second form, second shifting theorem.
$= e^{-\pi s}\mathcal{L}(\sin(t+\pi))$	Substitute $a = \pi$.
$= e^{-\pi s}\mathcal{L}(-\sin t)$	Sum rule $\sin(a+b) = \sin a \cos b + \sin b \cos a$ plus $\sin \pi = 0$, $\cos \pi = -1$.
$= e^{-\pi s}\dfrac{-1}{s^2+1}$	Basic Laplace table.
$=$ RHS	Identity verified.

15 Example (Trigonometric formulas) Show the steps used to obtain these Laplace identities:

(a) $\mathcal{L}(t\cos at) = \dfrac{s^2 - a^2}{(s^2+a^2)^2}$ **(c)** $\mathcal{L}(t^2\cos at) = \dfrac{2(s^3 - 3sa^2)}{(s^2+a^2)^3}$

(b) $\mathcal{L}(t\sin at) = \dfrac{2sa}{(s^2+a^2)^2}$ **(d)** $\mathcal{L}(t^2\sin at) = \dfrac{6s^2a - a^3}{(s^2+a^2)^3}$

Solution: The details for **(a)**:

$$\begin{aligned}\mathcal{L}(t\cos at) &= -(d/ds)\mathcal{L}(\cos at) && \text{Use } s\text{-differentiation.}\\ &= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) && \text{Basic Laplace table.}\\ &= \frac{s^2-a^2}{(s^2+a^2)^2} && \text{Calculus quotient rule.}\end{aligned}$$

The details for **(c)**:

$$\begin{aligned}\mathcal{L}(t^2\cos at) &= -(d/ds)\mathcal{L}((-t)\cos at) && \text{Use } s\text{-differentiation.}\\ &= \frac{d}{ds}\left(-\frac{s^2-a^2}{(s^2+a^2)^2}\right) && \text{Result of (a).}\\ &= \frac{2s^3-6sa^2)}{(s^2+a^2)^3} && \text{Calculus quotient rule.}\end{aligned}$$

The similar details for **(b)** and **(d)** are left as exercises.

16 Example (Exponentials) Show the steps used to obtain these Laplace identities:

(a) $\mathcal{L}(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$ **(c)** $\mathcal{L}(te^{at}\cos bt) = \dfrac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$

(b) $\mathcal{L}(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$ **(d)** $\mathcal{L}(te^{at}\sin bt) = \dfrac{2b(s-a)}{((s-a)^2+b^2)^2}$

Solution: Details for **(a)**:

$$\begin{aligned}\mathcal{L}(e^{at}\cos bt) &= \mathcal{L}(\cos bt)|_{s\to s-a} && \text{First shifting rule.}\\ &= \left.\left(\frac{s}{s^2+b^2}\right)\right|_{s\to s-a} && \text{Basic Laplace table.}\\ &= \frac{s-a}{(s-a)^2+b^2} && \text{Verified (a).}\end{aligned}$$

Details for **(c)**:

$$\begin{aligned}\mathcal{L}(te^{at}\cos bt) &= \mathcal{L}(t\cos bt)|_{s\to s-a} && \text{First shifting rule.}\\ &= \left.\left(-\frac{d}{ds}\mathcal{L}(\cos bt)\right)\right|_{s\to s-a} && \text{Apply } s\text{-differentiation.}\\ &= \left.\left(-\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)\right)\right|_{s\to s-a} && \text{Basic Laplace table.}\end{aligned}$$

$$= \left(\frac{s^2-b^2}{(s^2+b^2)^2}\right)\bigg|_{s\to s-a}$$ Calculus quotient rule.

$$= \frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$$ Verified **(c)**.

Left as exercises are **(b)** and **(d)**.

17 Example (Hyperbolic functions) Establish these Laplace transform facts about $\cosh u = (e^u + e^{-u})/2$ and $\sinh u = (e^u - e^{-u})/2$.

(a) $\mathcal{L}(\cosh at) = \dfrac{s}{s^2-a^2}$

(b) $\mathcal{L}(\sinh at) = \dfrac{a}{s^2-a^2}$

(c) $\mathcal{L}(t\cosh at) = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

(d) $\mathcal{L}(t\sinh at) = \dfrac{2as}{(s^2-a^2)^2}$

Solution: The details for **(a)**:

$$\mathcal{L}(\cosh at) = \tfrac{1}{2}(\mathcal{L}(e^{at}) + \mathcal{L}(e^{-at}))$$ Definition plus linearity of $\mathcal{L}$.

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$ Basic Laplace table.

$$= \frac{s}{s^2-a^2}$$ Identity **(a)** verified.

The details for **(d)**:

$$\mathcal{L}(t\sinh at) = -\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right)$$ Apply the s-differentiation rule.

$$= \frac{a(2s)}{(s^2-a^2)^2}$$ Calculus power rule; **(d)** verified.

Left as exercises are **(b)** and **(c)**.

18 Example (s-differentiation) Solve $\mathcal{L}(f(t)) = \dfrac{2s}{(s^2+1)^2}$ for $f(t)$.

Solution: The solution is $f(t) = t\sin t$. The details:

$$\mathcal{L}(f(t)) = \frac{2s}{(s^2+1)^2}$$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$ Calculus power rule $(u^n)' = nu^{n-1}u'$.

$$= -\frac{d}{ds}(\mathcal{L}(\sin t))$$ Basic Laplace table.

$$= \mathcal{L}(t\sin t)$$ Apply the s-differentiation rule.

$$f(t) = t\sin t$$ Lerch's cancellation law.

19 Example (First shift rule) Solve $\mathcal{L}(f(t)) = \dfrac{s+2}{2^2+2s+2}$ for $f(t)$.

Solution: The answer is $f(t) = e^{-t}\cos t + e^{-t}\sin t$. The details:

$\mathcal{L}(f(t)) = \dfrac{s+2}{s^2+2s+2}$	Signal for this method: the denominator has complex roots.
$= \dfrac{s+2}{(s+1)^2+1}$	Complete the square, denominator.
$= \dfrac{S+1}{S^2+1}$	Substitute S for $s+1$.
$= \dfrac{S}{S^2+1} + \dfrac{1}{S^2+1}$	Split into Laplace table entries.
$= \mathcal{L}(\cos t) + \mathcal{L}(\sin t)\big\|_{s\to S=s+1}$	Basic Laplace table.
$= \mathcal{L}(e^{-t}\cos t) + \mathcal{L}(e^{-t}\sin t)$	First shift rule.
$f(t) = e^{-t}\cos t + e^{-t}\sin t$	Invoke Lerch's cancellation law.

20 Example (Damped oscillator) Solve by Laplace's method the initial value problem $x'' + 2x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$.

Solution: The solution is $x(t) = e^{-t}\cos t$. The details:

$\mathcal{L}(x'') + 2\mathcal{L}(x') + 2\mathcal{L}(x) = \mathcal{L}(0)$	Apply $\mathcal{L}$ across the equation.
$s\mathcal{L}(x') - x'(0) + 2\mathcal{L}(x') + 2\mathcal{L}(x) = 0$	The t-derivative rule on x'.
$s[s\mathcal{L}(x) - x(0)] - x'(0)$ $+2[\mathcal{L}(x) - x(0)] + 2\mathcal{L}(x) = 0$	The t-derivative rule on x.
$(s^2+2s+2)\mathcal{L}(x) = 1 + s$	Use $x(0) = 1$, $x'(0) = -1$.
$\mathcal{L}(x) = \dfrac{s+1}{s^2+2s+2}$	Divide to isolate $\mathcal{L}(x)$.
$= \dfrac{s+1}{(s+1)^2+1}$	Complete the square in the denominator.
$= \mathcal{L}(\cos t)\big\|_{s\to s+1}$	Basic Laplace table.
$= \mathcal{L}(e^{-t}\cos t)$	First shifting rule.
$x(t) = e^{-t}\cos t$	Invoke Lerch's cancellation law.

21 Example (Rectified sine wave) Compute the Laplace transform of the rectified sine wave $f(t) = |\sin\omega t|$ and show that it can be expressed in the form

$$\mathcal{L}(|\sin\omega t|) = \frac{\omega\,\coth\left(\frac{\pi s}{2\omega}\right)}{s^2+\omega^2}.$$

Solution: The periodic function formula will be applied with period $P = 2\pi/\omega$. The calculation reduces to the evaluation of $J = \int_0^P f(t)e^{-st}dt$. Because $\sin\omega t \le 0$ on $\pi/\omega \le t \le 2\pi/\omega$, integral J can be written as $J = J_1 + J_2$, where

$$J_1 = \int_0^{\pi/\omega} \sin\omega t\, e^{-st}dt, \quad J_2 = \int_{\pi/\omega}^{2\pi/\omega} -\sin\omega t\, e^{-st}dt.$$

Integral tables give the result

$$\int \sin\omega t\, e^{-st}\, dt = -\frac{\omega e^{-st}\cos(\omega t)}{s^2+\omega^2} - \frac{s e^{-st}\sin(\omega t)}{s^2+\omega^2}.$$

Then

$$J_1 = \frac{\omega(e^{-\pi * s/\omega}+1)}{s^2+\omega^2}, \quad J_2 = \frac{\omega(e^{-2\pi s/\omega}+e^{-\pi s/\omega})}{s^2+\omega^2},$$

$$J = \frac{\omega(e^{-\pi s/\omega}+1)^2}{s^2+\omega^2}.$$

The remaining challenge is to write the answer for $\mathcal{L}(f(t))$ in terms of coth. The details:

$\mathcal{L}(f(t)) = \dfrac{J}{1-e^{-Ps}}$	Periodic function formula.
$= \dfrac{J}{(1-e^{-Ps/2})(1+e^{-Ps/2})}$	Apply $1-x^2=(1-x)(1+x)$, $x=e^{-Ps/2}$.
$= \dfrac{\omega(1+e^{-Ps/2})}{(1-e^{-Ps/2})(s^2+\omega^2)}$	Cancel factor $1+e^{-Ps/2}$.
$= \dfrac{e^{Ps/4}+e^{-Ps/4}}{e^{Ps/4}-e^{-Ps/4}}\dfrac{\omega}{s^2+\omega^2}$	Factor out $e^{-Ps/4}$, then cancel.
$= \dfrac{2\cosh(Ps/4)}{2\sinh(Ps/4)}\dfrac{\omega}{s^2+\omega^2}$	Apply cosh, sinh identities.
$= \dfrac{\omega\coth(Ps/4)}{s^2+\omega^2}$	Use $\coth u = \cosh u/\sinh u$.
$= \dfrac{\omega\coth\left(\frac{\pi s}{2\omega}\right)}{s^2+\omega^2}$	Identity verified.

22 Example (Half–wave Rectification) Determine the Laplace transform of the half–wave rectification of $\sin\omega t$, denoted $g(t)$, in which the negative cycles of $\sin\omega t$ have been replaced by zero, to create $g(t)$. Show in particular that

$$\mathcal{L}(g(t)) = \frac{1}{2}\frac{\omega}{s^2+\omega^2}\left(1+\coth\left(\frac{\pi s}{2\omega}\right)\right)$$

Solution: The half–wave rectification of $\sin\omega t$ is $g(t) = (\sin\omega t + |\sin\omega t|)/2$. Therefore, the basic Laplace table plus the result of Example 21 give

$$\begin{aligned}\mathcal{L}(2g(t)) &= \mathcal{L}(\sin\omega t) + \mathcal{L}(|\sin\omega t|)\\ &= \frac{\omega}{s^2+\omega^2} + \frac{\omega\cosh(\pi s/(2\omega))}{s^2+\omega^2}\\ &= \frac{\omega}{s^2+\omega^2}(1+\cosh(\pi s/(2\omega))\end{aligned}$$

Dividing by 2 produces the identity.

23 Example (Shifting Rules I) Solve $\mathcal{L}(f(t)) = e^{-3s}\dfrac{s+1}{s^2+2s+2}$ for $f(t)$.

Solution: The answer is $f(t) = e^{3-t}\cos(t-3)H(t-3)$. The details:

$$\begin{aligned}
\mathcal{L}(f(t)) &= e^{-3s}\frac{s+1}{(s+1)^2+1} && \text{Complete the square.}\\
&= e^{-3s}\frac{S}{S^2+1} && \text{Replace } s+1 \text{ by } S.\\
&= e^{-3S+3}\left(\mathcal{L}(\cos t)\right)\big|_{s\to S=s+1} && \text{Basic Laplace table.}\\
&= e^{3}\left(e^{-3s}\mathcal{L}(\cos t)\right)\big|_{s\to S=s+1} && \text{Regroup factor } e^{-3S}.\\
&= e^{3}\left(\mathcal{L}(\cos(t-3)H(t-3))\right)\big|_{s\to S=s+1} && \text{Second shifting rule.}\\
&= e^{3}\mathcal{L}(e^{-t}\cos(t-3)H(t-3)) && \text{First shifting rule.}\\
f(t) &= e^{3-t}\cos(t-3)H(t-3) && \text{Lerch's cancellation law.}
\end{aligned}$$

24 Example (Shifting Rules II) Solve $\mathcal{L}(f(t) = \dfrac{s+7}{s^2+4s+8}$ for $f(t)$.

Solution: The answer is $f(t) = e^{-2t}(\cos 2t + \frac{5}{2}\sin 2t)$. The details:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \frac{s+7}{(s+2)^2+4} && \text{Complete the square.}\\
&= \frac{S+5}{S^2+4} && \text{Replace } s+2 \text{ by } S.\\
&= \frac{S}{S^2+4} + \frac{5}{2}\frac{2}{S^2+4} && \text{Split into table entries.}\\
&= \left.\frac{s}{s^2+4} + \frac{5}{2}\frac{2}{s^2+4}\right|_{s\to S=s+2} && \text{Shifting rule preparation.}\\
&= \mathcal{L}\left(\cos 2t + \tfrac{5}{2}\sin 2t\right)\big|_{s\to S=s+2} && \text{Basic Laplace table.}\\
&= \mathcal{L}(e^{-2t}(\cos 2t + \tfrac{5}{2}\sin 2t)) && \text{First shifting rule.}\\
f(t) &= e^{-2t}(\cos 2t + \tfrac{5}{2}\sin 2t) && \text{Lerch's cancellation law.}
\end{aligned}$$

Exercises 8.3

Second Order Initial Value Problems. Display the Laplace method details which verify the supplied answer.

1. $x'' + x = 1$, $x(0) = 1$, $x'(0) = 0$; $x(t) = 1$.

2. $x'' + 4x = 4$, $x(0) = 1$, $x'(0) = 0$; $x(t) = 1$.

3. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 1$; $x(t) = \cos t + \sin t$.

4. $x'' + x = 0$, $x(0) = 1$, $x'(0) = 2$; $x(t) = \cos t + 2\sin t$.

5. $x'' + 2x' + x = 0$, $x(0) = 0$, $x'(0) = 1$; $x(t) = te^{-t}$.

6. $x'' + 2x' + x = 0$, $x(0) = 1$, $x'(0) = -1$; $x(t) = e^{-t}$.

7. $x'' + 3x' + 2x = 0$, $x(0) = 1$, $x'(0) = -1$; $x(t) = e^{-t}$.

8. $x'' + 3x' + 2x = 0$, $x(0) = 1$, $x'(0) = -2$; $x(t) = e^{-2t}$.

9. $x'' + 3x' = 0$, $x(0) = 5$, $x'(0) = 0$; $x(t) = 5$.

10. $x'' + 3x' = 0$, $x(0) = 1$, $x'(0) = -3$; $x(t) = e^{-3t}$.

11. $x'' = 2$, $x(0) = 0$, $x'(0) = 0$; $x(t) = t^2$.

12. $x'' = 6t$, $x(0) = 0$, $x'(0) = 0$; $x(t) = t^3$.

13. $x'' = 2$, $x(0) = 0$, $x'(0) = 1$; $x(t) = t + t^2$.

14. $x'' = 6t$, $x(0) = 0$, $x'(0) = 1$; $x(t) = t + t^3$.

15. $x'' + x' = 6t$, $x(0) = 0$, $x'(0) = 1$; $x(t) = t + t^3$.

16. $x'' + x' = 6t$, $x(0) = 0$, $x'(0) = 1$; $x(t) = t + t^3$.

8.4 Heaviside's Method

The method solves an equation like

$$\mathcal{L}(f(t)) = \frac{2s}{(s+1)(s^2+1)}$$

for the t-expression $f(t) = -e^{-t} + \cos t + \sin t$. The details in Heaviside's method involve a sequence of easy-to-learn college algebra steps. This practical method was popularized by the English electrical engineer Oliver Heaviside (1850–1925).

More precisely, **Heaviside's method** starts with a polynomial quotient

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} \tag{1}$$

and computes an expression $f(t)$ such that

$$\frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \mathcal{L}(f(t)) \equiv \int_0^\infty f(t)e^{-st}dt.$$

It is assumed that $a_0, \ldots, a_n, b_0, \ldots, b_m$ are constants and the polynomial quotient (1) has limit zero at $s = \infty$.

Partial Fraction Theory

In college algebra, it is shown that a rational function (1) can be expressed as the sum of **partial fractions**.,

Definition 2 (Partial Fraction)
A partial fraction is a polynomial fraction with a constant in the numerator and a polynomial denominator having exactly one root.

By definition, a partial fraction has the form

$$\frac{A}{(s-s_0)^k}. \tag{2}$$

The numerator in (2) is a real or complex constant A. The denominator has exactly one root $s = s_0$, real or complex. The power $(s - s_0)^k$ **must divide the denominator** in (1).

Assume fraction (1) has **real coefficients**. If s_0 in (2) is real, then A is *real*. If $s_0 = \alpha + i\beta$ in (2) is *complex*, then $(s - \overline{s_0})^k$ also divides the denominator in (1), where $\overline{s_0} = \alpha - i\beta$ is the complex conjugate of s_0. The corresponding partial fractions used in the expansion turn out to be complex conjugates of one another, which can be paired and re-written as a fraction

$$\frac{A}{(s-s_0)^k} + \frac{\overline{A}}{(s-\overline{s_0})^k} = \frac{Q(s)}{((s-\alpha)^2+\beta^2)^k}, \tag{3}$$

where $Q(s)$ is a *real* polynomial.

To illustrate, if $A = c + id$, then

$$\begin{aligned}\frac{A}{(s-2i)^2} + \frac{\overline{A}}{(s+2i)^2} &= \frac{(A+\overline{A})s^2 + 4i(\overline{A}-A)s - 4(A+\overline{A})}{(s^2+4)^2} \\ &= \frac{2cs^2 + 8ds - 8c}{(s^2+4)^2}.\end{aligned}$$

The numerator can be expanded as $Q(s) = A_1(s^2+4) + B_1 s + C_1$, with real coefficients A_1, B_1, C_1. Then the fraction can be written as

$$\frac{Q(s)}{(s^2+4)^2} = \frac{A_1}{s^2+4} + \frac{B_1 s + C_1}{(s^2+4)^2}.$$

This justifies the formal replacement of all partial fractions with denominator $(s-s_0)^k$ or $(s-\overline{s_0})^k$, s_0 complex, by

$$\frac{B + Cs}{((s-s_0)(s-\overline{s_0}))^k} = \frac{B+Cs}{((s-\alpha)^2+\beta^2)^k},$$

in which B and C are real constants. This **real form** is preferred over the sum of complex fractions, because integral tables and Laplace tables typically contain only real formulas.

Simple Roots. Assume that (1) has *real coefficients* and the denominator of the fraction (1) has **distinct real roots** $s_1, \ldots, s_N$ and **distinct complex roots** $\alpha_1 \pm i\beta_1, \ldots, \alpha_M \pm i\beta_M$. The partial fraction expansion of (1) is a sum given in terms of *real* constants A_p, B_q, C_q by

$$\text{(4)}\qquad \frac{a_0 + a_1 s + \cdots + a_n s^n}{b_0 + b_1 s + \cdots + b_m s^m} = \sum_{p=1}^{N} \frac{A_p}{s-s_p} + \sum_{q=1}^{M} \frac{B_q + C_q(s-\alpha_q)}{(s-\alpha_q)^2 + \beta_q^2}.$$

Multiple Roots. Assume (1) has *real coefficients* and the denominator of the fraction (1) has possibly **multiple roots**. Let N_p be the multiplicity of real root s_p and let M_q be the multiplicity of complex root $\alpha_q + i\beta_q$ ($\beta_q > 0$), $1 \le p \le N$, $1 \le q \le M$. The partial fraction expansion of (1) is given in terms of *real* constants $A_{p,k}$, $B_{q,k}$, $C_{q,k}$ by

$$\text{(5)}\qquad \sum_{p=1}^{N} \sum_{1\le k\le N_p} \frac{A_{p,k}}{(s-s_p)^k} + \sum_{q=1}^{M} \sum_{1\le k\le M_q} \frac{B_{q,k} + C_{q,k}(s-\alpha_q)}{((s-\alpha_q)^2+\beta_q^2)^k}.$$

Summary. The theory for simple roots and multiple roots can be distilled as follows.

A polynomial quotient p/q with limit zero at infinity has a unique expansion into partial fractions. A partial fraction is either a constant divided by a divisor of q having exactly one real root, or else a linear function divided by a real divisor of q, having exactly one complex conjugate pair of roots.

The Sampling Method

Consider the expansion in partial fractions

$$\frac{s-1}{s(s+1)^2(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{Ds+E}{s^2+1}. \tag{6}$$

The five undetermined real constants A through E are found by **clearing the fractions**, that is, multiply (6) by the denominator on the left to obtain the polynomial equation

$$\begin{aligned} s-1 &= A(s+1)^2(s^2+1) + Bs(s+1)(s^2+1) \\ &\quad + Cs(s^2+1) + (Ds+E)s(s+1)^2. \end{aligned} \tag{7}$$

Next, five different **samples** of s are substituted into (7) to obtain equations for the five unknowns A through E.[2] Always use the **roots of the denominator** to start: $s=0$, $s=-1$, $s=i$, $s=-i$ are the roots of $s(s+1)^2(s^2+1)=0$. Each complex root results in two equations, by taking real and imaginary parts. The complex conjugate root $s=-i$ is not used, because it duplicates equations already obtained from $s=i$. The three roots $s=0$, $s=-1$, $s=i$ give only four equations, so we **invent another sample** $s=1$ to get the fifth equation:

$$\begin{aligned} -1 &= A & (s=0) \\ -2 &= -2C & (s=-1) \\ i-1 &= (Di+E)i(i+1)^2 & (s=i) \\ 0 &= 8A+4B+2C+4(D+E) & (s=1) \end{aligned} \tag{8}$$

Because D and E are real, the complex equation $(s=i)$ becomes two equations, as follows.

$i-1=(Di+E)i(i^2+2i+1)$	Expand power.
$i-1=-2Di-2E$	Simplify using $i^2=-1$.
$1=-2D$	Equate imaginary parts.
$-1=-2E$	Equate real parts.

Solving the 5×5 system, the answers are $A=-1$, $B=3/2$, $C=1$, $D=-1/2$, $E=1/2$.

[2] The values chosen for s are called **samples**, that is, cleverly chosen values. The number of s-values sampled equals the number of symbols A, B, ... to be determined.

The Method of Atoms

Consider the expansion in partial fractions

$$(9) \qquad \frac{2s-2}{s(s+1)^2(s^2+1)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{(s+1)^2} + \frac{ds+e}{s^2+1}.$$

Clearing the fractions in (9) gives the polynomial equation

$$(10) \qquad \begin{aligned} 2s-2 &= a(s+1)^2(s^2+1) + bs(s+1)(s^2+1) \\ &\quad + cs(s^2+1) + (ds+e)s(s+1)^2. \end{aligned}$$

The **method of atoms** expands all polynomial products and collects on powers of s (functions 1, s, s^2, ... are by *definition* called Euler solution **atoms**). The coefficients of the powers are matched to give 5 equations in the five unknowns a through e. Some details:

$$(11) \qquad \begin{aligned} 2s-2 &= (a+b+d)\, s^4 + (2a+b+c+2d+e)\, s^3 \\ &\quad + (2a+b+d+2e)\, s^2 + (2a+b+c+e)\, s + a \end{aligned}$$

Matching powers of s implies the 5 equations

$$\begin{aligned} &a+b+d=0,\ 2a+b+c+2d+e=0,\ 2a+b+d+2e=0, \\ &2a+b+c+e=2,\ a=-2. \end{aligned}$$

Solving, the unique solution is $a=-2$, $b=3$, $c=2$, $d=-1$, $e=1$.

Heaviside's Coverup Method

The method applies only to the case of distinct roots of the denominator in (1). Extensions to multiple-root cases can be made; see page 602.

To illustrate Oliver Heaviside's ideas, consider the problem details

$$(12) \qquad \begin{aligned} \frac{2s+1}{s(s-1)(s+1)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1} \\ &= \mathcal{L}(A) + \mathcal{L}(Be^t) + \mathcal{L}(Ce^{-t}) \\ &= \mathcal{L}(A + Be^t + Ce^{-t}) \end{aligned}$$

The first line (12) uses college algebra partial fractions. The second and third lines use the basic Laplace table and linearity of $\mathcal{L}$.

Mysterious Details. Oliver Heaviside proposed to find in (12) the constant $C = -\frac{1}{2}$ by a **cover–up method**:

$$\left.\frac{2s+1}{s(s-1)\boxed{}}\right|_{\boxed{s+1}=0} = \frac{C}{\boxed{}}.$$

The *instructions* are to cover–up the matching factors $(s+1)$ on the left and right with box $\boxed{}$ (Heaviside used his fingertips), then evaluate on the left at the *root* s which causes the box contents to be zero. The other terms on the right are replaced by zero.

To justify Heaviside's cover–up method, **clear the fraction** $C/(s+1)$, that is, multiply (12) by the denominator $\boxed{s+1}$ of the partial fraction $C/(s+1)$ to obtain the *partially-cleared fraction relation*

$$\frac{(2s+1)\boxed{(s+1)}}{s(s-1)\boxed{(s+1)}} = \frac{A\boxed{(s+1)}}{s} + \frac{B\boxed{(s+1)}}{s-1} + \frac{C\boxed{(s+1)}}{\boxed{(s+1)}}.$$

Set $\boxed{(s+1)} = 0$ in the display. Cancelations left and right plus annihilation of two terms on the right gives Heaviside's prescription

$$\left.\frac{2s+1}{s(s-1)}\right|_{\boxed{s+1}=0} = C.$$

The factor $(s+1)$ in (12) is by no means special: the same procedure applies to find A and B. The method works for denominators with simple roots, that is, no repeated roots are allowed.

Heaviside's method in words:[3]

> To determine A in a given partial fraction $\frac{A}{s-s_0}$, multiply the relation by $(s-s_0)$, which partially clears the fraction. Substitute s from the equation $s - s_0 = 0$ into the partially cleared relation.

Extension to Multiple Roots. Heaviside's method can be extended to the case of repeated roots. The basic idea is to *factor–out the repeats.* To illustrate, consider the partial fraction expansion details

$R = \dfrac{1}{(s+1)^2(s+2)}$	A sample rational function having repeated roots.
$= \dfrac{1}{s+1}\left(\dfrac{1}{(s+1)(s+2)}\right)$	Factor–out the repeats.
$= \dfrac{1}{s+1}\left(\dfrac{1}{s+1} + \dfrac{-1}{s+2}\right)$	Apply the cover–up method to the simple root fraction.
$= \dfrac{1}{(s+1)^2} + \dfrac{-1}{(s+1)(s+2)}$	Multiply.
$= \dfrac{1}{(s+1)^2} + \dfrac{-1}{s+1} + \dfrac{1}{s+2}$	Apply the cover–up method to the last fraction on the right.

[3]Root $s = s_0$ is called a **pole** and the answer A is called a **residue**. See page 604.

Terms with only one root in the denominator are already partial fractions. Thus the work centers on expansion of quotients in which the denominator has two or more roots.

Special Methods. Heaviside's method has a useful extension for the case of roots of multiplicity two. To illustrate, consider these details:

$R = \dfrac{1}{(s+1)^2(s+2)}$	$\boxed{1}$ A fraction with multiple roots.
$= \dfrac{A}{s+1} + \dfrac{B}{(s+1)^2} + \dfrac{C}{s+2}$	$\boxed{2}$ See equation (5), page 599.
$= \dfrac{A}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{s+2}$	$\boxed{3}$ Find B and C by Heaviside's cover–up method.
$= \dfrac{-1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{s+2}$	$\boxed{4}$ Details below.

We discuss $\boxed{4}$ details. Multiply the equation $\boxed{1} = \boxed{2}$ by $s+1$ to partially clear fractions, the same step as the cover-up method:

$$\frac{1}{(s+1)(s+2)} = A + \frac{B}{s+1} + \frac{C(s+1)}{s+2}.$$

We don't substitute $s + 1 = 0$, because it gives infinity for the second term. Instead, set $s = \infty$ to get the equation $0 = A + C$. Because $C = 1$ from $\boxed{3}$, then $A = -1$.

The illustration works for one root of multiplicity two, because $s = \infty$ will resolve the coefficient not found by the cover–up method.

In general, if the denominator in (1) has a root s_0 of multiplicity k, then the partial fraction expansion contains terms

$$\frac{A_1}{s-s_0} + \frac{A_2}{(s-s_0)^2} + \cdots + \frac{A_k}{(s-s_0)^k}.$$

Heaviside's cover–up method directly finds A_k, but not A_1 to A_{k-1}.

Cover-up Method and Complex Numbers. Consider the partial fraction expansion

$$\frac{10}{(s+1)(s^2+9)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+9}.$$

The symbols A, B, C are real. The value of A can be found directly by the cover-up method, giving $A = 1$. To find B and C, multiply the fraction expansion by s^2+9, in order to partially clear fractions, then formally set $s^2+9=0$ to obtain the two equations

$$\frac{10}{s+1} = Bs + C, \quad s^2+9=0.$$

The method applies the identical idea used for one real root. By clearing fractions in the first, the equations become

$$10 = Bs^2 + Cs + Bs + C, \quad s^2+9=0.$$

Substitute $s^2=-9$ into the first equation to give the linear equation

$$10 = (-9B + C) + (B + C)s.$$

Because this linear equation has two complex roots $s = \pm 3i$, then real constants B, C satisfy the 2×2 system

$$\begin{array}{rcrcr} -9B & + & C & = & 10, \\ B & + & C & = & 0. \end{array}$$

Solving gives $B=-1$, $C=1$.

The same method applies especially to fractions with 3-term denominators, like s^2+s+1. The only change made in the details is the replacement $s^2 \to -s-1$. By repeated application of $s^2=-s-1$, the first equation can be distilled into one linear equation in s with two roots. As before, a 2×2 system results.

Residues, Poles and Oliver Heaviside

The language of **residues** and **poles** invaded engineering literature years ago, blamed in part on engineers who studied the foundations of complex variables. The terminology formalizes the naming of partial fraction theory constants and roots that appear in Oliver Heaviside's *cover-up method*, an electrical engineering partial fraction shortcut that dates back to the year 1890.

Residues and poles do not provide any new mathematical tools for solving partial fraction problems. The service of residues and poles is to provide a new language for discussing the answers, a language that appears in current engineering and science literature. If you know how to compute coefficients in partial fractions using Heaviside's shortcut, then you already know how to find residues and poles.

An Example. Heaviside's shortcut finds the coefficients $c_1 = \frac{1}{2}, c_2 = -5, c_3 = \frac{5}{2}$ in the expansion

$$\frac{5-(2(s+2)(s+3)}{(s+1)(s+2)(s+3)} = \frac{c_1}{s+1} + \frac{c_2}{s+2} + \frac{c_3}{s+3}$$

by clearing the fractions one at a time, each clearing followed by substitution of the corresponding root found in the denominator.

For example, to clear the fraction for c_2 requires multiplication by $(s+2)$, to give the intermediate step (Heaviside did it mentally, writing nothing)

$$\frac{5-(2(s+2)(s+3)}{(s+1)(s+3)} = \frac{c_1(s+2)}{s+1} + \frac{c_2}{1} + \frac{c_3(s+2)}{s+3}.$$

The root of $s+2=0$ is then substituted to give $c_2=-5$.

Table 8. Working Definition of Pole and Residue

A **pole** is the same as a root of the denominator in a quotient $\dfrac{p(x)}{q(x)}$.
A **residue** is the same as a coefficient in the partial fraction expansion of the quotient $\dfrac{p(x)}{q(x)}$ (precise details below).

In the example, the **residue** at **pole** $s=-2$ (the root of $s+2=0$) is **defined** by the equation

$$\lim_{s\to-2}(s+2)\frac{5-(2(s+2)(s+3)}{(s+1)(s+2)(s+3)}.$$

To evaluate the limit, cancel the common factor $(s+2)$ and substitute $s=-2$. Oliver Heaviside would be surprised by the unnecessary limit.

Definition 3 (Poles and Residues)
A function $f(z)$ of complex variable z has a **pole** at $z=z_0$ provided there is an integer $n\ge 0$ such that $g(z)=(z-z_0)^n f(z)$ can be written as a power series

$$g(z) = g_0 + g_1(z-z_0) + g_2(z-z_0)^2 + \cdots$$

convergent in a disk $|z-z_0|<R$ and $g_0\ne 0$ (which means $g(z_0)\ne 0$).

The **order of the pole** is the integer n. The **residue** is g_0.

If $f(z)$ has a pole $z=z_0$ of order n, then the **residue** g_0 at the pole can be computed from the limit formula

$$g_0 = \lim_{z\to z_0}(z-z_0)^n f(z).$$

In terms of series expansion, a pole of order n means that

$$f(z) = \frac{g_0}{(z-z_0)^n} + \cdots + g_n + g_{n+1}(z-z_0) + g_{n+2}(z-z_0)^2 + \cdots,$$

which is called a **Laurent Series**.

Table 9. Pole, Residue and Applications

A real pole defines the **damping coefficient** in a transient.

A complex pole on the imaginary axis describes **frequency**.

Residues are **mode shape** information.

Examples

25 **Example (Partial Fractions I)** Show the details of the partial fraction expansion

$$\frac{s^3+2s^2+2s+5}{(s-1)(s^2+4)(s^2+2s+2)} = \frac{2/5}{s-1} + \frac{1/2}{s^2+4} - \frac{1}{10}\frac{7+4\,s}{s^2+2\,s+2}.$$

Solution:

Background. The problem originates as equality $\boxed{5}=\boxed{6}$ in the sequence of Example 27, page 609, which solves for $x(t)$ using the method of partial fractions:

$$\boxed{5} \qquad \mathcal{L}(x) = \frac{s^3+2s^2+2s+5}{(s-1)(s^2+4)(s^2+2s+2)}$$

$$\boxed{6} \qquad = \frac{2/5}{s-1} + \frac{1/2}{s^2+4} - \frac{1}{10}\frac{7+4\,s}{s^2+2\,s+2}$$

College algebra detail. College algebra partial fractions theory says that there exist real constants A, B, C, D, E satisfying the identity

$$\frac{s^3+2s^2+2s+5}{(s-1)(s^2+4)(s^2+2s+2)} = \frac{A}{s-1} + \frac{B+Cs}{s^2+4} + \frac{D+Es}{s^2+2\,s+2}.$$

As explained on page 598, the complex conjugate roots $\pm 2i$ and $-1 \pm i$ are not represented as terms $c/(s-s_0)$, but in the combined real form seen in the above display, which is suited for use with Laplace tables.

The **sampling method** applies to find the constants. In this method, the fractions are cleared to obtain the polynomial relation

$$\begin{aligned} s^3+2s^2+2s+5 \;=\;& A(s^2+4)(s^2+2s+2) \\ &+(B+Cs)(s-1)(s^2+2s+2) \\ &+(D+Es)(s-1)(s^2+4). \end{aligned}$$

The roots of the denominator $(s-1)(s^2+4)(s^2+2s+2)$ to be inserted into the previous equation are $s=1$, $s=2i$, $s=-1+i$. The conjugate roots $s=-2i$ and $s=-1-i$ are not used. Each complex root generates two equations, by equating real and imaginary parts, therefore there will be 5 equations in 5 unknowns. Substitution of $s=1$, $s=2i$, $s=-1+i$ gives three equations

$$\begin{array}{lrcl} s=1 & 10 &=& 25A, \\ s=2i & -4i-3 &=& (B+2iC)(2i-1)(-4+4i+2), \\ s=-1+i & 5 &=& (D-E+Ei)(-2+i)(2-2(-1+i)). \end{array}$$

Writing each expanded complex equation in terms of its real and imaginary parts, explained in detail below, gives 5 equations

$$\begin{array}{llcrcr} s=1 & 2 & = & 5A, & & \\ s=2i & -3 & = & -6B & + & 16C, \\ s=2i & -4 & = & -8B & - & 12C, \\ s=-1+i & 5 & = & -6D & - & 2E, \\ s=-1+i & 0 & = & 8D & - & 14E. \end{array}$$

The equations are solved to give $A = 2/5$, $B = 1/2$, $C = 0$, $D = -7/10$, $E = -2/5$ (details for B, C below).

Complex equation to two real equations. It is an algebraic mystery how exactly the complex equation

$$-4i - 3 = (B + 2iC)(2i - 1)(-4 + 4i + 2)$$

gets converted into two real equations. The process is explained here.

First, the complex equation is expanded, as though it is a polynomial in variable i, to give the steps

$$\begin{array}{rcll} -4i-3 & = & (B+2iC)(2i-1)(-2+4i) & \\ & = & (B+2iC)(-4i+2+8i^2-4i) & \text{Expand.} \\ & = & (B+2iC)(-6-8i) & \text{Use } i^2=-1. \\ & = & -6B-12iC-8Bi+16C & \text{Expand, use } i^2=-1. \\ & = & (-6B+16C)+(-8B-12C)i & \text{Convert to form } x+yi. \end{array}$$

Next, the two sides are compared. Because B and C are real, then the real part of the right side is $(-6B + 16C)$ and the imaginary part of the right side is $(-8B - 12C)$. Equating matching parts on each side gives the equations

$$\begin{array}{rcl} -6B+16C & = & -3, \\ -8B-12C & = & -4, \end{array}$$

which is a 2×2 linear system for the unknowns B, C.

Solving the 2×2 system. Such a system with a unique solution can be solved by Cramer's rule, matrix inversion or elimination. The answer: $B = 1/2$, $C = 0$.

The easiest method turns out to be elimination. Multiply the first equation by 4 and the second equation by 3, then subtract to obtain $C = 0$. Then the first equation is $-6B + 0 = -3$, implying $B = 1/2$.

26 Example (Partial Fractions II) Verify the partial fraction expansion

$$\frac{s^5+8\,s^4+23\,s^3+31\,s^2+24\,s+9}{(s+1)^2\,(s^2+s+1)^2} = \frac{4}{s+1} + \frac{5-3s}{s^2+s+1}.$$

Solution:

Basic partial fraction theory implies that there are unique real constants a, b, c, d, e, f satisfying the equation

$$(13)\quad \begin{array}{rcl} \dfrac{s^5+8\,s^4+23\,s^3+31\,s^2+24\,s+9}{(s+1)^2\,(s^2+s+1)^2} & = & \dfrac{a}{s+1}+\dfrac{b}{(s+1)^2} \\ & & +\dfrac{c+ds}{s^2+s+1}+\dfrac{e+f\,s}{(s^2+s+1)^2} \end{array}$$

The **sampling** method applies to clear fractions and replace the fractional equation by the polynomial relation

$$\begin{aligned} s^5+8\,s^4+23\,s^3+31\,s^2+24\,s+9 &= a(s+1)(s^2+s+1)^2 \\ &\quad +b(s^2+s+1)^2 \\ &\quad +(c+ds)(s^2+s+1)(s+1)^2 \\ &\quad +(e+f\,s)(s+1)^2 \end{aligned}$$

However, the prognosis for the resultant algebra is grim: only three of the six required equations can be obtained by substitution of the roots ($s=-1$, $s=-1/2+i\sqrt{3}/2$) of the denominator. We abandon the idea, because of the complexity of the 6×6 system of linear equations required to solve for the constants a through f.

Instead, the fraction R on the left of (13) is written with repeated factors extracted, as follows:

$$\begin{aligned} R &= \frac{1}{(s+1)(s^2+s+1)}\left(\frac{p(x)}{(s+1)(s^2+s+1)}\right), \\ p(x) &= s^5+8\,s^4+23\,s^3+31\,s^2+24\,s+9. \end{aligned}$$

Long division gives the formula

$$\frac{p(x)}{(s+1)(s^2+s+1)} = s^2+6s+9.$$

Therefore, the fraction R on the left of (13) can be written as

$$R = \frac{p(x)}{(s+1)^2(s^2+s+1)^2} = \frac{(s+3)^2}{(s+1)(s^2+s+1)}.$$

The simplified form of R has a partial fraction expansion

$$\frac{(s+3)^2}{(s+1)(s^2+s+1)} = \frac{a}{s+1}+\frac{b+cs}{s^2+s+1}.$$

Heaviside's cover-up method gives $a=4$. Applying Heaviside's method again to the quadratic factor implies the pair of equations

$$\frac{(s+3)^2}{s+1} = b+cs, \quad s^2+s+1=0.$$

These equations can be solved for $b=5$, $c=-3$. The details assume that s is a root of $s^2+s+1=0$, then

$\dfrac{(s+3)^2}{s+1} = b+cs$	The first equation.
$\dfrac{s^2+6s+9}{s+1} = b+cs$	Expand.
$\dfrac{-s-1+6s+9}{s+1} = b+cs$	Use $s^2+s+1=0$.
$5s+8=(s+1)(b+cs)$	Clear fractions.
$5s+8=bs+cs+b+cs^2$	Expand again.
$5s+8=bs+cs+b-cs-c$	Use $s^2+s+1=0$.

The conclusion $5 = b$ and $8 = b - c$ follows because the last equation is linear but has two complex roots. Then $b = 5$, $c = -3$.

27 Example (Third Order Initial Value Problem) Solve the third order initial value problem

$$x''' - x'' + 4x' - 4x = 5e^{-t}\sin t,$$
$$x(0) = 0, \quad x'(0) = x''(0) = 1.$$

Solution:
The answer is

$$x(t) = \frac{2}{5}e^{t} + \frac{1}{4}\sin 2t - \frac{3}{10}e^{-t}\sin t - \frac{2}{5}e^{-t}\cos t.$$

Method. Apply $\mathcal{L}$ to the differential equation. In steps $\boxed{1}$ to $\boxed{3}$ the Laplace integral of $x(t)$ is isolated, by applying linearity of $\mathcal{L}$, integration by parts $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ and the basic Laplace table.

$$\mathcal{L}(x''') - \mathcal{L}(x'') + 4\mathcal{L}(x') - 4\mathcal{L}(x) = 5\mathcal{L}(e^{-t}\sin t) \qquad \boxed{1}$$

$$(s^3\mathcal{L}(x) - s - 1) - (s^2\mathcal{L}(x) - 1) + 4(s\mathcal{L}(x)) - 4\mathcal{L}(x) = \frac{5}{(s+1)^2+1} \qquad \boxed{2}$$

$$(s^3 - s^2 + 4s - 4)\mathcal{L}(x) = 5\frac{1}{(s+1)^2+1} + s \qquad \boxed{3}$$

Steps $\boxed{5}$ and $\boxed{6}$ use the college algebra theory of partial fractions, the details of which appear in Example 25, page 606. Steps $\boxed{7}$ and $\boxed{8}$ write the partial fraction expansion in terms of Laplace table entries. Step $\boxed{9}$ converts the s-expressions, which are basic Laplace table entries, into Laplace integral expressions. Algebraically, we replace s-expressions by expressions in symbols $\mathcal{L}$ and t.

$$\begin{aligned}
\mathcal{L}(x) &= \frac{\frac{5}{(s+1)^2+1} + s}{s^3 - s^2 + 4s - 4} && \boxed{4}\\
&= \frac{s^3 + 2s^2 + 2s + 5}{(s-1)(s^2+4)(s^2+2s+2)} && \boxed{5}\\
&= \frac{2/5}{s-1} + \frac{1/2}{s^2+4} - 1/10\,\frac{7+4\,s}{s^2+2\,s+2} && \boxed{6}\\
&= \frac{2/5}{s-1} + \frac{1/2}{s^2+4} - 1/10\,\frac{3+4(s+1)}{(s+1)^2+1} && \boxed{7}\\
&= \frac{2/5}{s-1} + \frac{1/2}{s^2+4} - \frac{3/10}{(s+1)^2+1} - \frac{(2/5)(s+1)}{(s+1)^2+1} && \boxed{8}\\
&= \mathcal{L}\left(\frac{2}{5}e^{t} + \frac{1}{4}\sin 2t - \frac{3}{10}e^{-t}\sin t - \frac{2}{5}e^{-t}\cos t\right) && \boxed{9}
\end{aligned}$$

The last step $\boxed{10}$ applies Lerch's cancelation theorem to the equation $\boxed{4}=\boxed{9}$.

$$x(t) = \frac{2}{5}e^{t} + \frac{1}{4}\sin 2t - \frac{3}{10}e^{-t}\sin t - \frac{2}{5}e^{-t}\cos t \qquad \boxed{10}$$

28 Example (Second Order System) Solve for $x(t)$ and $y(t)$ in the 2nd order system of linear differential equations

$$\begin{array}{ll} 2x'' - x' + 9x - y'' - y' - 3y = 0, & x(0) = x'(0) = 1, \\ 2'' + x' + 7x - y'' + y' - 5y = 0, & y(0) = y'(0) = 0. \end{array}$$

Solution: The answer is

$$\begin{aligned} x(t) &= \frac{1}{3}e^t + \frac{2}{3}\cos(2\,t) + \frac{1}{3}\sin(2\,t), \\ y(t) &= \frac{2}{3}e^t - \frac{2}{3}\cos(2\,t) - \frac{1}{3}\sin(2\,t). \end{aligned}$$

Transform. The intent of steps $\boxed{1}$ and $\boxed{2}$ is to transform the initial value problem into two equations in two unknowns. Used repeatedly in $\boxed{1}$ is integration by parts $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$. No Laplace tables were used. In $\boxed{2}$ the substitutions $x_1 = \mathcal{L}(x)$, $x_2 = \mathcal{L}(y)$ are made to produce two equations in the two unknowns x_1, x_2.

$$\begin{array}{l} (2s^2 - s + 9)\mathcal{L}(x) + (-s^2 - s - 3)\mathcal{L}(y) = 1 + 2s, \\ (2s^2 + s + 7)\mathcal{L}(x) + (-s^2 + s - 5)\mathcal{L}(y) = 3 + 2s, \end{array} \qquad \boxed{1}$$

$$\begin{array}{lllll} (2s^2 - s + 9)x_1 & + & (-s^2 - s - 3)x_2 & = & 1 + 2s, \\ (2s^2 + s + 7)x_1 & + & (-s^2 + s - 5)x_2 & = & 3 + 2s. \end{array} \qquad \boxed{2}$$

Step $\boxed{3}$ uses Cramer's rule to compute the answers x_1, x_2 to the equations $ax_1 + bx_2 = e$, $cx_1 + dx_2 = f$ as the determinant fractions

$$x_1 = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.$$

The variable names x_1, x_2 stand for the Laplace integrals of the unknowns $x(t)$, $y(t)$, respectively. The answers, following a calculation:

$$\begin{cases} x_1 = \dfrac{s^2 + 2/3}{s^3 - s^2 + 4\,s - 4}, \\ x_2 = \dfrac{10/3}{s^3 - s^2 + 4\,s - 4}. \end{cases} \qquad \boxed{3}$$

Step $\boxed{4}$ writes each fraction resulting from Cramer's rule as a partial fraction expansion suited for reverse Laplace table look-up. Step $\boxed{5}$ does the table look-up and prepares for step $\boxed{6}$ to apply Lerch's cancelation law, in order to display the answers $x(t)$, $y(t)$.

$$\begin{cases} x_1 = \dfrac{1/3}{s-1} + \dfrac{2}{3}\dfrac{s}{s^2+4} + \dfrac{1}{3}\dfrac{2}{s^2+4}, \\ x_2 = \dfrac{2/3}{s-1} - \dfrac{2}{3}\dfrac{s}{s^2+4} - \dfrac{1}{3}\dfrac{2}{s^2+4}. \end{cases} \qquad \boxed{4}$$

$$\begin{cases} \mathcal{L}(x(t)) = \mathcal{L}\left(\frac{1}{3}e^t + \frac{2}{3}\cos(2\,t) + \frac{1}{3}\sin(2\,t)\right), \\ \mathcal{L}(y(t)) = \mathcal{L}\left(\frac{2}{3}e^t - \frac{2}{3}\cos(2\,t) - \frac{1}{3}\sin(2\,t)\right). \end{cases} \quad \boxed{5}$$

$$\begin{cases} x(t) = \frac{1}{3}e^t + \frac{2}{3}\cos(2\,t) + \frac{1}{3}\sin(2\,t), \\ y(t) = \frac{2}{3}e^t - \frac{2}{3}\cos(2\,t) - \frac{1}{3}\sin(2\,t). \end{cases} \quad \boxed{6}$$

Partial fraction details. We will show how to obtain the expansion

$$\frac{s^2+2/3}{s^3-s^2+4\,s-4} = \frac{1/3}{s-1} + \frac{2}{3}\,\frac{s}{s^2+4} + \frac{1}{3}\,\frac{2}{s^2+4}.$$

The denominator $s^3 - s^2 + 4\,s - 4$ factors into $s-1$ times s^2+4. Partial fraction theory implies that there is an expansion with *real coefficients* A, B, C of the form

$$\frac{s^2+2/3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}.$$

We will verify $A = 1/3$, $B = 2/3$, $C = 2/3$. Clear the fractions to obtain the polynomial equation

$$s^2 + 2/3 = A(s^2+4) + (Bs+C)(s-1).$$

Instead of using $s = 1$ and $s = 2i$, which are roots of the denominator, we shall use $s = 1$, $s = 0$, $s = -1$ to get a *real* 3×3 system for variables A, B, C:

$$\begin{array}{lrcl} s=1: & 1+2/3 & = & A(1+4)+0, \\ s=0: & 0+2/3 & = & A(4)+C(-1), \\ s=-1: & 1+2/3 & = & A(1+4)+(-B+C)(-2). \end{array}$$

Write this system as an augmented matrix G with variables A, B, C assigned to the first three columns of G:

$$G = \left(\begin{array}{ccr|c} 5 & 0 & 0 & 5/3 \\ 4 & 0 & -1 & 2/3 \\ 5 & 2 & -2 & 5/3 \end{array}\right)$$

Using computer assist, calculate

$$\mathbf{rref}(G) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 2/3 \\ 0 & 0 & 1 & 2/3 \end{array}\right)$$

Then A, B, C are the last column entries of $\mathbf{rref}(G)$, which verifies the partial fraction expansion.

Heaviside cover-up detail. It is possible to rapidly check that $A = 1/3$ using the cover-up method. Less obvious is that the cover-up method also applies to the fraction with complex roots.

The idea is to multiply the fraction decomposition by $s^2 + 4$ to partially clear the fractions and then set $s^2 + 4 = 0$. This process formally sets s equal to one of the two roots $s = \pm 2i$. We avoid complex numbers entirely by solving for B, C in the pair of equations

$$\frac{s^2 + 2/3}{s - 1} = A(0) + (Bs + C), \quad s^2 + 4 = 0.$$

Because $s^2 = -4$, the first equality is simplified to $\dfrac{-4 + 2/3}{s - 1} = Bs + C$. Swap sides of the equation, then cross-multiply to obtain $Bs^2 + Cs - Bs - C = -10/3$ and then use $s^2 = -4$ again to simplify to $(-B + C)s + (-4B - C) = -10/3$. Because this linear equation in variable s has two solutions, then $-B + C = 0$ and $-4B - C = -10/3$. Solve this 2×2 system by elimination to obtain $B = C = 2/3$.

We review the algebraic method. First, we found two equations in symbols s, B, C. Next, symbol s is eliminated to give two equations in symbols B, C. Finally, the 2×2 system for B, C is solved.

8.5 Transform Properties

Collected here are the major theorems for the manipulation of Laplace transform tables, along with their derivations. Students who study in isolation are advised to dwell on the details of proof and re-read the examples of preceding sections.

Theorem 5 (Linearity)
The Laplace transform has these inherited integral properties:

$$\begin{array}{ll} \textbf{(a)} & \mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t)), \\ \textbf{(b)} & \mathcal{L}(cf(t)) = c\mathcal{L}(f(t)). \end{array}$$

Theorem 6 (The t-Derivative Rule)
Let $y(t)$ be continuous, of exponential order and let $y'(t)$ be piecewise continuous on $t \ge 0$. Then $\mathcal{L}(y'(t))$ exists and

$$\mathcal{L}(y'(t)) = s\mathcal{L}(y(t)) - y(0).$$

Theorem 7 (The t-Integral Rule)
Let $g(t)$ be of exponential order and continuous for $t \ge 0$. Then

$$\mathcal{L}\left(\int_0^t g(x)\,dx\right) = \frac{1}{s}\mathcal{L}(g(t))$$

or equivalently

$$\mathcal{L}(g(t)) = s\mathcal{L}\left(\int_0^t g(x)\,dx\right)$$

Theorem 8 (The s-Differentiation Rule)
Let $f(t)$ be of exponential order. Then

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t)).$$

Theorem 9 (First Shifting Rule)
Let $f(t)$ be of exponential order and $-\infty < a < \infty$. Then

$$\mathcal{L}(e^{at}f(t)) = \left.\mathcal{L}(f(t))\right|_{s\to(s-a)}.$$

Theorem 10 (Second Shifting Rule)
Let $f(t)$ and $g(t)$ be of exponential order and assume $a \ge 0$. Then

$$\begin{array}{ll} \textbf{(a)} & \mathcal{L}(f(t-a)H(t-a)) = e^{-as}\mathcal{L}(f(t)), \\ \textbf{(b)} & \mathcal{L}(g(t)H(t-a)) = e^{-as}\mathcal{L}(g(t+a)). \end{array}$$

Theorem 11 (Periodic Function Rule)
Let $f(t)$ be of exponential order and satisfy $f(t+P) = f(t)$. Then

$$\mathcal{L}(f(t)) = \frac{\int_0^P f(t)e^{-st}dt}{1-e^{-Ps}}.$$

Theorem 12 (Convolution Rule)
Let $f(t)$ and $g(t)$ be of exponential order. Then

$$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \mathcal{L}\left(\int_0^t f(x)g(t-x)dx\right).$$

Theorem 13 (Initial and Final Value Rules)
Let $f(t)$ and $f'(t)$ be functions of exponential order. Then, when all indicated limits exist,

$$\begin{array}{llllll} \mathbf{1}. & f(\infty) & = & \lim_{t\to\infty} f(t) & = & \lim_{s\to 0} s\mathcal{L}(f(t)), \\ \mathbf{2}. & f(0+) & = & \lim_{t\to 0+} f(t) & = & \lim_{s\to\infty} s\mathcal{L}(f(t)), \\ \mathbf{3}. & & & \lim_{s\to\infty} \mathcal{L}(f(t)) & = & 0. \end{array}$$

Applying Theorem 13. Impulses and higher order singularities at $t = 0$ are disallowed, because hypotheses imply $s\mathcal{L}(f(t))$ bounded. Examples $f(t) = \sin t$ and $f(t) = e^t$ don't satisfy hypotheses, so **2** is inapplicable. Denominator roots of $s\mathcal{L}(f(t))$ should not be in the right half-plane.

DC Gain and the Final Value Theorem

The three parameters $\zeta, \omega, G_{\mathbf{DC}}$ appearing in the **underdamped** model

$$x'' + 2\zeta\omega x' + \omega^2 x = G_{\mathbf{DC}}\,\omega^2\,\mathbf{u}(t)$$

are known respectively as the damping ratio, frequency and DC-gain. Under-damped means $\zeta > 1$. Symbol $\mathbf{u}(t)$ is the unit step function.

Superposition implies that $x = x_h + x_p$ is decomposed with equilibrium solution $x_p(t) = G_{\mathbf{DC}}$ and homogeneous general solution $x_h(t) = c_1 e^{-at}\cos(bt) + c_2 e^{-at}\sin(bt)$, where $a = \zeta\omega$ and $b = \omega\sqrt{\zeta^2 - 1}$. Because of the exponential decay of $x_h(t)$, the constant solution $x_p(t) = G_{\mathbf{DC}}$ is an equilibrium state.

The DC gain is formally equal to the limit of the output/input at $t = \infty$. In a lab, it is found from a constant input k, then the DC gain is the steady state of the output divided by k.

The *final value theorem* can be used to find the DC gain $G_{\mathbf{DC}}$. Assume the input is a multiple of the unit step function, $k\,\mathbf{u}(t)$, then find the final value, which is the steady state of the output/input, equal to the DC gain $G_{\mathbf{DC}} = k/\omega^2$ of the system.

Background. The **DC Gain** in an audio amplifier equals 1 when 0 dB input has 0 dB output, an ideal situation the best amplifiers realize approximately. Gain may be voltage gain (OP-amp, V/V), power gain (RF-amp, W/W) or sensor gain (light, e.g., 5 μV per photon). The acronym DC means direct current, no derivatives or integrals in the equations, steady-state with all transients removed.

Proofs and Details

Proof of Theorem 5 (Linearity):

LHS $= \mathcal{L}(f(t)+g(t))$	Left side of the identity in (a).
$= \int_0^\infty (f(t)+g(t))e^{-st}dt$	Direct transform.
$= \int_0^\infty f(t)e^{-st}dt + \int_0^\infty g(t)e^{-st}dt$	Calculus integral rule.
$= \mathcal{L}(f(t)) + \mathcal{L}(g(t))$	Equals RHS; identity (a) verified.
LHS $= \mathcal{L}(cf(t))$	Left side of the identity in (b).
$= \int_0^\infty cf(t)e^{-st}dt$	Direct transform.
$= c\int_0^\infty f(t)e^{-st}dt$	Calculus integral rule.
$= c\mathcal{L}(f(t))$	Equals RHS; identity (b) verified.

Proof of Theorem 6 (t-Derivative rule): Already $\mathcal{L}(f(t))$ exists, because f is of exponential order and continuous. On an interval $[a,b]$ where f' is continuous, integration by parts using $u = e^{-st}$, $dv = f'(t)dt$ gives

$$\begin{aligned}\int_a^b f'(t)e^{-st}dt &= f(t)e^{-st}\Big|_{t=a}^{t=b} - \int_a^b f(t)(-s)e^{-st}dt \\ &= -f(a)e^{-sa} + f(b)e^{-sb} + s\int_a^b f(t)e^{-st}dt.\end{aligned}$$

On any interval $[0,N]$, there are finitely many intervals $[a,b]$ on each of which f' is continuous. Add the above equality across these finitely many intervals $[a,b]$. The boundary values on adjacent intervals match and the integrals add to give

$$\int_0^N f'(t)e^{-st}dt = -f(0)e^0 + f(N)e^{-sN} + s\int_0^N f(t)e^{-st}dt.$$

Take the limit across this equality as $N \to \infty$. Then the right side has limit $-f(0) + s\mathcal{L}(f(t))$, because of the existence of $\mathcal{L}(f(t))$ and $\lim_{t\to\infty} f(t)e^{-st} = 0$ for large s. Therefore, the left side has a limit, and by definition $\mathcal{L}(f'(t))$ exists and $\mathcal{L}(f'(t)) = -f(0) + s\mathcal{L}(f(t))$.

Proof of Theorem 7 (t-Integral rule): Let $f(t) = \int_0^t g(x)dx$. Then f is of exponential order and continuous. The details:

$\mathcal{L}(\int_0^t g(x)dx) = \mathcal{L}(f(t))$	By definition.
$= \frac{1}{s}\mathcal{L}(f'(t))$	Because $f(0)=0$ implies $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t))$.
$= \frac{1}{s}\mathcal{L}(g(t))$	Because $f' = g$ by the Fundamental theorem of calculus.

Proof of Theorem 8 (s-Differentiation): We prove the equivalent relation $\mathcal{L}((-t)f(t)) = (d/ds)\mathcal{L}(f(t))$. If f is of exponential order, then so is $(-t)f(t)$, therefore $\mathcal{L}((-t)f(t))$ exists. It remains to show the s-derivative exists and satisfies the given equality.

The proof below is based in part upon the calculus inequality

$$(1) \qquad \left|e^{-x} + x - 1\right| \le x^2, \quad x \ge 0.$$

The inequality is obtained from two applications of the *mean value theorem* $g(b)-g(a) = g'(\overline{x})(b-a)$, which gives $e^{-x}+x-1 = x\overline{x}e^{-x_1}$ with $0 \le x_1 \le \overline{x} \le x$.

In addition, the existence of $\mathcal{L}(t^2|f(t)|)$ is used to define $s_0 > 0$ such that $\mathcal{L}(t^2|f(t)|) \le 1$ for $s > s_0$. This follows from the transform existence theorem for functions of exponential order, where it is shown that the transform has limit zero at $s = \infty$.

Consider $h \neq 0$ and the Newton quotient $Q(s,h) = (F(s+h) - F(s))/h$ for the s-derivative of the Laplace integral. We have to show that

$$\lim_{h\to 0} |Q(s,h) - \mathcal{L}((-t)f(t))| = 0.$$

This will be accomplished by proving for $s > s_0$ and $s+h > s_0$ the inequality

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h|.$$

For $h \neq 0$,

$$Q(s,h) - \mathcal{L}((-t)f(t)) = \int_0^\infty f(t)\frac{e^{-st-ht} - e^{-st} + the^{-st}}{h}\,dt.$$

Assume $h > 0$. Due to the exponential rule $e^{A+B} = e^A e^B$, the quotient in the integrand simplifies to give

$$Q(s,h) - \mathcal{L}((-t)f(t)) = \int_0^\infty f(t)e^{-st}\left(\frac{e^{-ht} + th - 1}{h}\right) dt.$$

Inequality (1) applies with $x = ht \ge 0$, giving

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h| \int_0^\infty t^2|f(t)|e^{-st}dt.$$

The right side is $|h|\mathcal{L}(t^2|f(t)|)$, which for $s > s_0$ is bounded by $|h|$, completing the proof for $h > 0$. If $h < 0$, then a similar calculation is made to obtain

$$|Q(s,h) - \mathcal{L}((-t)f(t))| \le |h| \int_0^\infty t^2|f(t)e^{-st-ht}dt.$$

The right side is $|h|\mathcal{L}(t^2|f(t)|)$ evaluated at $s+h$ instead of s. If $s+h > s_0$, then the right side is bounded by $|h|$, completing the proof for $h < 0$.

Proof of Theorem 9 (First Shifting Rule): The left side LHS of the equality can be written because of the exponential rule $e^A e^B = e^{A+B}$ as

$$\text{LHS} = \int_0^\infty f(t)e^{-(s-a)t}dt.$$

This integral is $\mathcal{L}(f(t))$ with s replaced by $s-a$, which is precisely the meaning of the right side RHS of the equality. Therefore, LHS = RHS.

Proof of Theorem 10 (Second Shifting Rule): The details for (a) are

$$\begin{aligned} \text{LHS} &= \mathcal{L}(H(t-a)f(t-a)) \\ &= \textstyle\int_0^\infty H(t-a)f(t-a)e^{-st}dt \quad \text{Direct transform.} \end{aligned}$$

$$
\begin{aligned}
&= \int_a^\infty H(t-a)f(t-a)e^{-st}dt && \text{Because } a \geq 0 \text{ and } H(x)=0 \text{ for } x<0.\\
&= \int_0^\infty H(x)f(x)e^{-s(x+a)}dx && \text{Change variables } x=t-a,\ dx=dt.\\
&= e^{-sa}\int_0^\infty f(x)e^{-sx}dx && \text{Use } H(x)=1 \text{ for } x\geq 0.\\
&= e^{-sa}\mathcal{L}(f(t)) && \text{Direct transform.}\\
&= \text{RHS} && \text{Identity (a) verified.}
\end{aligned}
$$

In the details for (b), let $f(t)=g(t+a)$, then

$$
\begin{aligned}
\text{LHS} &= \mathcal{L}(H(t-a)g(t))\\
&= \mathcal{L}(H(t-a)f(t-a)) && \text{Use } f(t-a)=g(t-a+a)=g(t).\\
&= e^{-sa}\mathcal{L}(f(t)) && \text{Apply (a).}\\
&= e^{-sa}\mathcal{L}(g(t+a)) && \text{Because } f(t)=g(t+a).\\
&= \text{RHS} && \text{Identity (b) verified.}
\end{aligned}
$$

Proof of Theorem 11 (Periodic Function Rule):

$$
\begin{aligned}
\text{LHS} &= \mathcal{L}(f(t))\\
&= \int_0^\infty f(t)e^{-st}dt && \text{Direct transform.}\\
&= \sum_{n=0}^\infty \int_{nP}^{nP+P} f(t)e^{-st}dt && \text{Additivity of the integral.}\\
&= \sum_{n=0}^\infty \int_0^P f(x+nP)e^{-sx-nPs}dx && \text{Change variables } t=x+nP.\\
&= \sum_{n=0}^\infty e^{-nPs}\int_0^P f(x)e^{-sx}dx && \text{Because } f(x) \text{ is } P\text{–periodic and } e^Ae^B=e^{A+B}.\\
&= \int_0^P f(x)e^{-sx}dx \sum_{n=0}^\infty r^n && \text{The summation has a common factor. Define } r=e^{-Ps}.\\
&= \int_0^P f(x)e^{-sx}dx \,\frac{1}{1-r} && \text{Sum the geometric series.}\\
&= \frac{\int_0^P f(x)e^{-sx}dx}{1-e^{-Ps}} && \text{Substitute } r=e^{-Ps}.\\
&= \text{RHS} && \text{Periodic function identity verified.}
\end{aligned}
$$

Left unmentioned here is the convergence of the infinite series on line 3 of the proof, which follows from f of exponential order.

Proof of Theorem 12 (Convolution rule): The details use Fubini's integration interchange theorem for a planar unbounded region, and therefore this proof involves advanced calculus methods that may be outside the background of the reader. Modern calculus texts contain a less general version of Fubini's theorem for finite regions, usually referenced as *iterated integrals*. The unbounded planar region is written in two ways:

$$
\begin{aligned}
D &= \{(r,t) : t \leq r < \infty,\ 0 \leq t < \infty\},\\
\mathcal{D} &= \{(r,t) : 0 \leq r < \infty,\ 0 \leq r \leq t\}.
\end{aligned}
$$

Readers should pause here and verify that $D=\mathcal{D}$.

The change of variable $r = x + t$, $dr = dx$ is applied for fixed $t \geq 0$ to obtain the identity

$$(2) \qquad \begin{aligned} e^{-st}\int_0^\infty g(x)e^{-sx}dx &= \int_0^\infty g(x)e^{-sx-st}dx \\ &= \int_t^\infty g(r-t)e^{-rs}dr. \end{aligned}$$

The left side of the convolution identity is expanded as follows:

$\mathsf{LHS} = \mathcal{L}(f(t))\mathcal{L}(g(t))$	
$= \int_0^\infty f(t)e^{-st}dt \int_0^\infty g(x)e^{-sx}dx$	Direct transform.
$= \int_0^\infty f(t)\int_t^\infty g(r-t)e^{-rs}drdt$	Apply identity (2).
$= \int_D f(t)g(r-t)e^{-rs}drdt$	Fubini's theorem applied.
$= \int_{\mathcal{D}} f(t)g(r-t)e^{-rs}drdt$	Descriptions D and $\mathcal{D}$ are the same.
$= \int_0^\infty \int_0^r f(t)g(r-t)dte^{-rs}dr$	Fubini's theorem applied.

Then

$\mathsf{RHS} = \mathcal{L}\left(\int_0^t f(u)g(t-u)du\right)$	
$= \int_0^\infty \int_0^t f(u)g(t-u)due^{-st}dt$	Direct transform.
$= \int_0^\infty \int_0^r f(u)g(r-u)due^{-sr}dr$	Change variable names $r \leftrightarrow t$.
$= \int_0^\infty \int_0^r f(t)g(r-t)dt\, e^{-sr}dr$	Change variable names $u \leftrightarrow t$.
$= \mathsf{LHS}$	Convolution identity verified.

8.6 Heaviside Step and Dirac Impulse

Heaviside Function. The **unit step function** $\mathbf{u}(t)$ and its more precise clone the **Heaviside function**[4] $H(t)$ are defined by

$$\mathbf{u}(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 1 & \text{for} \quad t = 0, \\ 0 & \text{for} \quad t < 0, \end{cases} \qquad H(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ \text{undefined} & \text{for} \quad t = 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

The most often–used formula involving the unit step function is the **characteristic function** of the interval $a \le t < b$, or **pulse**, given by

$$\begin{aligned} \mathbf{pulse}(t, a, b) &= \mathbf{u}(t-a) - \mathbf{u}(t-b) \\ &= \begin{cases} 1 & a \le t < b, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

To illustrate, a square wave $\textsf{sqw}(t) = (-1)^{\textsf{floor}(t)}$ can be written in the series form

$$\sum_{n=0}^{\infty} (-1)^n \, \mathbf{pulse}(t, n, n+1).$$

In modern computer algebra systems like `maple`, there is a distinction between the piecewise-defined unit step function and the Heaviside function. The Heaviside function $H(t)$ is technically undefined at $t = 0$, whereas the unit step is defined everywhere. This seemingly minor distinction is more sensible when taking formal derivatives: dH/dt is zero except at $t = 0$ where it is undefined. The issue decided is the domain of dH/dt, which is all $t \ne 0$.

Dirac Impulse. A precise mathematical definition of the Dirac impulse, denoted δ, is not possible to give here. Following its inventor Paul Dirac, the definition should be

$$\delta(t) = \frac{d}{dt}\, \mathbf{u}(t).$$

The latter is nonsensical, because the unit step does not have a calculus derivative at $t = 0$. However, $d\,\mathbf{u}(t)$ could have the meaning of a Riemann-Stieltjes integrator, which restrains $d\,\mathbf{u}(t)$ to have meaning only under an integral sign. It is in this sense that the Dirac impulse δ is defined.

What do we mean by the differential equation

$$x'' + 16x = 5\delta(t - t_0)?$$

[4] A technical requirement may make the Heaviside function undefined at $t = 0$, which distinguishes it from the unit step function. All functions in Laplace theory are assumed zero for $t < 0$, therefore 1 and the unit step are identical for $t \ge 0$.

The equation $x'' + 16x = f(t)$ represents a spring-mass system without damping having Hooke's constant 16, subject to external force $f(t)$. In a mechanical context, the Dirac impulse term $5\delta(t - t_0)$ is an *idealization* of a hammer-hit at time $t = t_0 > 0$ with impulse 5.

More precisely, the forcing term $f(t)$ in $x'' + 16x = f(t)$ can be formally written as a Riemann-Stieltjes integrator $5d\,\mathbf{u}(t - t_0)$ where $\mathbf{u}$ is the unit step function. The Dirac impulse or *derivative of the unit step*, nonsensical as it may appear, is realized in applications via the two-sided or central difference quotient

$$\frac{\mathbf{u}(t+h) - \mathbf{u}(t-h)}{2h} \approx dH(t).$$

Therefore, the force $f(t)$ in the idealization $5\delta(t - t_0)$ is given for $h > 0$ very small by the approximation

$$f(t) \approx 5\frac{\mathbf{u}(t - t_0 + h) - \mathbf{u}(t - t_0 - h)}{2h}.$$

The *impulse*[5] of the approximated force over a large interval $[a, b]$ is computed from

$$\int_a^b f(t)dt \approx 5\int_{-h}^{h} \frac{\mathbf{u}(t - t_0 + h) - \mathbf{u}(t - t_0 - h)}{2h}dt = 5,$$

due to the integrand being $1/(2h)$ on $|t - t_0| < h$ and otherwise 0.

Modeling Impulses. One argument for the Dirac impulse idealization is that an infinity of choices exist for modeling an impulse. There are in addition to the central difference quotient two other popular difference quotients, the forward quotient $(\mathbf{u}(t + h) - \mathbf{u}(t))/h$ and the backward quotient $(\mathbf{u}(t) - \mathbf{u}(t - h))/h$ ($h > 0$ assumed). In reality, h is unknown in any application, and the impulsive force of a hammer hit is hardly constant, as is supposed by this naive modeling.

The modeling logic often applied for the Dirac impulse is that the external force $f(t)$ is used in the model in a limited manner, in which only the momentum $p = mv$ is important. More precisely, only the change in momentum or impulse is important, $\int_a^b f(t)dt = \Delta p = mv(b) - mv(a)$.

The precise force $f(t)$ is replaced during the modeling by a simplistic piecewise-defined force that has exactly the same impulse Δp. The replacement is justified by arguing that if only the impulse is important, and not the actual details of the force, then both models should give similar results.

[5]Momentum is defined to be mass times velocity. If the force f is given by Newton's law as $f(t) = \frac{d}{dt}(mv(t))$ and $v(t)$ is velocity, then $\int_a^b f(t)dt = mv(b) - mv(a)$ is the net momentum or impulse.

Function or Operator? The work of physics Nobel prize winner P. Dirac (1902–1984) proceeded for about 20 years before the mathematical community developed a sound mathematical theory for his impulsive force representations. A systematic theory was developed in 1936 by the Soviet mathematician S. Sobolev. The French mathematician L. Schwartz further developed the theory in 1945. He observed that the idealization is not a function but an operator or *linear functional*, in particular, δ maps or *associates* to each function $\phi(t)$ its value at $t = 0$, in short, $\delta(\phi) = \phi(0)$. This fact was observed early on by Dirac and others, during the replacement of simplistic forces by δ.

In Laplace theory, there is a natural encounter with the ideas, because $\mathcal{L}(f(t))$ routinely appears on the right of the equation after transformation. This term, in the case of an impulsive force $f(t) = c(H(t - t_0 - h) - H(t - t_0 + h))/(2h)$, evaluates for $t_0 > 0$ and $t_0 - h > 0$ as follows:

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty \frac{c}{2h}(H(t - t_0 - h) - H(t - t_0 + h))e^{-st}dt \\
&= \int_{t_0-h}^{t_0+h} \frac{c}{2h}e^{-st}dt \\
&= ce^{-st_0}\left(\frac{e^{sh} - e^{-sh}}{2sh}\right)
\end{aligned}$$

The factor $\dfrac{e^{sh} - e^{-sh}}{2sh}$ is approximately 1 for $h > 0$ small, because of L'Hôspital's rule. The immediate conclusion is that we should replace the impulsive force f by an equivalent one f^* such that

$$\mathcal{L}(f^*(t)) = ce^{-st_0}.$$

Unfortunately, *there is no such function* f^*!

The apparent mathematical flaw in this idea was resolved by the work of L. Schwartz on **distributions**. In short, there is a solid foundation for introducing f^*, but unfortunately the mathematics involved is not elementary nor especially accessible to those readers whose background is just calculus.

Practising engineers and scientists might be able to ignore the vast literature on distributions, citing the example of physicist P. Dirac, who succeeded in applying impulsive force ideas without the distribution theory developed by S. Sobolev and L. Schwartz. This will not be the case for those who wish to read current literature on partial differential equations, because the work on distributions has forever changed the required background for that topic.

8.7 Laplace Table Derivations

Verified here are two Laplace tables, the minimal Laplace Table 7.2-4 and its extension Table 7.2-5. Largely, this section is for reading, as it is designed to enrich lectures and to aid readers who study in isolation.

Derivation of Laplace integral formulas in Table 7.2-4, page 583.

- **Proof of** $\mathcal{L}(t^n) = n!/s^{1+n}$:

The first step is to evaluate $\mathcal{L}(t^n)$ for $n = 0$.

$\mathcal{L}(1) = \int_0^\infty (1)e^{-st}dt$	Laplace integral of $f(t) = 1$.
$= -(1/s)e^{-st}\big\vert_{t=0}^{t=\infty}$	Evaluate the integral.
$= 1/s$	Assumed $s > 0$ to evaluate $\lim_{t\to\infty} e^{-st}$.

The value of $\mathcal{L}(t^n)$ for $n = 1$ can be obtained by s-differentiation of the relation $\mathcal{L}(1) = 1/s$, as follows.

$\frac{d}{ds}\mathcal{L}(1) = \frac{d}{ds}\int_0^\infty (1)e^{-st}dt$	Laplace integral for $f(t) = 1$.
$= \int_0^\infty \frac{d}{ds}\left(e^{-st}\right)dt$	Used $\frac{d}{ds}\int_a^b F dt = \int_a^b \frac{dF}{ds}dt$.
$= \int_0^\infty (-t)e^{-st}dt$	Calculus rule $(e^u)' = u'e^u$.
$= -\mathcal{L}(t)$	Definition of $\mathcal{L}(t)$.

Then

$\mathcal{L}(t) = -\frac{d}{ds}\mathcal{L}(1)$	Rewrite last display.
$= -\frac{d}{ds}(1/s)$	Use $\mathcal{L}(1) = 1/s$.
$= 1/s^2$	Differentiate.

This idea can be repeated to give $\mathcal{L}(t^2) = -\frac{d}{ds}\mathcal{L}(t)$ and hence $\mathcal{L}(t^2) = 2/s^3$. The pattern is $\mathcal{L}(t^n) = -\frac{d}{ds}\mathcal{L}(t^{n-1})$ which gives $\mathcal{L}(t^n) = n!/s^{1+n}$.

- **Proof of** $\mathcal{L}(e^{at}) = 1/(s-a)$:

The result follows from $\mathcal{L}(1) = 1/s$, as follows.

$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}dt$	Direct Laplace transform.
$= \int_0^\infty e^{-(s-a)t}dt$	Use $e^Ae^B = e^{A+B}$.
$= \int_0^\infty e^{-St}dt$	Substitute $S = s - a$.
$= 1/S$	Apply $\mathcal{L}(1) = 1/s$.
$= 1/(s-a)$	Back-substitute $S = s - a$.

- **Proof of** $\mathcal{L}(\cos bt) = s/(s^s + b^2)$ **and** $\mathcal{L}(\sin bt) = b/(s^s + b^2)$:

Use will be made of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, usually first introduced in trigonometry. In this formula, θ is a real number in radians and $i = \sqrt{-1}$ is the complex unit.

$e^{ibt}e^{-st} = (\cos bt)e^{-st} + i(\sin bt)e^{-st}$	Substitute $\theta = bt$ into Euler's formula and multiply by e^{-st}.
$\int_0^\infty e^{-ibt}e^{-st}dt = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$	Integrate $t = 0$ to $t = \infty$. Then use properties of integrals.
$\dfrac{1}{s-ib} = \int_0^\infty (\cos bt)e^{-st}dt + i\int_0^\infty (\sin bt)e^{-st}dt$	Evaluate the left hand side using $\mathcal{L}(e^{at}) = 1/(s-a)$, $a = ib$.
$\dfrac{1}{s-ib} = \mathcal{L}(\cos bt) + i\mathcal{L}(\sin bt)$	Direct Laplace transform definition.
$\dfrac{s+ib}{s^2+b^2} = \mathcal{L}(\cos bt) + i\mathcal{L}(\sin bt)$	Use complex rule $1/z = \overline{z}/\|z\|^2$, $z = A+iB$, $\overline{z} = A-iB$, $\|z\| = \sqrt{A^2+B^2}$.
$\dfrac{s}{s^2+b^2} = \mathcal{L}(\cos bt)$	Extract the real part.
$\dfrac{b}{s^2+b^2} = \mathcal{L}(\sin bt)$	Extract the imaginary part.

Derivation of Laplace integral formulas in Table 7.2-5, page 583.

- **Proof of the Heaviside formula** $\mathcal{L}(H(t-a)) = e^{-as}/s$.

$\mathcal{L}(H(t-a)) = \int_0^\infty H(t-a)e^{-st}dt$	Direct Laplace transform. Assume $a \ge 0$.
$= \int_a^\infty (1)e^{-st}dt$	Because $H(t-a) = 0$ for $0 \le t < a$.
$= \int_0^\infty (1)e^{-s(x+a)}dx$	Change variables $t = x + a$.
$= e^{-as}\int_0^\infty (1)e^{-sx}dx$	Constant e^{-as} moves outside integral.
$= e^{-as}(1/s)$	Apply $\mathcal{L}(1) = 1/s$.

- **Proof of the Dirac impulse formula** $\mathcal{L}(\delta(t-a)) = e^{-as}$.

The *definition* of the Dirac impulse is a formal one, in which every occurrence of $\delta(t-a)dt$ under an integrand is replaced by $dH(t-a)$. The differential symbol $dH(t-a)$ is taken in the sense of the Riemann-Stieltjes integral. This integral is defined in [?] for monotonic integrators $\alpha(x)$ as the limit

$$\int_a^b f(x)d\alpha(x) = \lim_{N\to\infty}\sum_{n=1}^{N} f(x_n)(\alpha(x_n) - \alpha(x_{n-1}))$$

where $x_0 = a$, $x_N = b$ and $x_0 < x_1 < \cdots < x_N$ forms a partition of $[a, b]$ whose mesh approaches zero as $N \to \infty$.

The steps in computing the Laplace integral of the Dirac impulse appear below. Admittedly, the proof requires advanced calculus skills and a certain level of mathematical maturity. The reward is a fuller understanding of the Dirac symbol $\delta(x)$.

$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)dt$	Laplace integral, $a > 0$ assumed.
$= \int_0^\infty e^{-st}dH(t-a)$	Replace $\delta(t-a)dt$ by $dH(t-a)$.
$= \lim_{M\to\infty}\int_0^M e^{-st}dH(t-a)$	Definition of improper integral.

$= e^{-sa}$ Explained below.

To explain the last step, apply the definition of the Riemann-Stieltjes integral:

$$\int_0^M e^{-st} dH(t-a) = \lim_{N\to\infty} \sum_{n=0}^{N-1} e^{-st_n}(H(t_n - a) - H(t_{n-1} - a))$$

where $0 = t_0 < t_1 < \cdots < t_N = M$ is a partition of $[0, M]$ whose mesh $\max_{1\le n\le N}(t_n - t_{n-1})$ approaches zero as $N \to \infty$. Given a partition, if $t_{n-1} < a \le t_n$, then $H(t_n - a) - H(t_{n-1} - a) = 1$, otherwise this factor is zero. Therefore, the sum reduces to a single term e^{-st_n}. This term approaches e^{-sa} as $N \to \infty$, because t_n must approach a.

- **Proof of** $\mathcal{L}(\textbf{floor}(t/a)) = \dfrac{e^{-as}}{s(1 - e^{-as})}$:

The library function **floor** present in computer languages C and Fortran is defined by $\textbf{floor}(x)$ = greatest whole integer $\le x$, e.g., $\textbf{floor}(5.2) = 5$ and $\textbf{floor}(-1.9) = -2$. The computation of the Laplace integral of $\textbf{floor}(t)$ requires ideas from infinite series, as follows.

$F(s) = \int_0^\infty \textbf{floor}(t) e^{-st} dt$	Laplace integral definition.
$= \sum_{n=0}^\infty \int_n^{n+1} (n) e^{-st} dt$	On $n \le t < n+1$, $\textbf{floor}(t) = n$.
$= \sum_{n=0}^\infty \dfrac{n}{s}(e^{-ns} - e^{-ns-s})$	Evaluate each integral.
$= \dfrac{1 - e^{-s}}{s} \sum_{n=0}^\infty n e^{-sn}$	Common factor removed.
$= \dfrac{x(1-x)}{s} \sum_{n=0}^\infty n x^{n-1}$	Define $x = e^{-s}$.
$= \dfrac{x(1-x)}{s} \dfrac{d}{dx} \sum_{n=0}^\infty x^n$	Term-by-term differentiation.
$= \dfrac{x(1-x)}{s} \dfrac{d}{dx} \dfrac{1}{1-x}$	Geometric series sum.
$= \dfrac{x}{s(1-x)}$	Compute the derivative, simplify.
$= \dfrac{e^{-s}}{s(1 - e^{-s})}$	Substitute $x = e^{-s}$.

To evaluate the Laplace integral of $\textbf{floor}(t/a)$, a change of variables is made.

$\mathcal{L}(\textbf{floor}(t/a)) = \int_0^\infty \textbf{floor}(t/a) e^{-st} dt$	Laplace integral definition.
$= a \int_0^\infty \textbf{floor}(r) e^{-asr} dr$	Change variables $t = ar$.
$= aF(as)$	Apply the formula for $F(s)$.
$= \dfrac{e^{-as}}{s(1 - e^{-as})}$	Simplify.

- **Proof of** $\mathcal{L}(\textbf{sqw}(t/a)) = \dfrac{1}{s}\tanh(as/2)$:

The square wave defined by $\textbf{sqw}(x) = (-1)^{\textbf{floor}(x)}$ is periodic of period 2 and piecewise-defined. Let $\mathcal{P} = \int_0^2 \textbf{sqw}(t) e^{-st} dt$.

$$\begin{aligned}
\mathcal{P} &= \int_0^1 \textsf{sqw}(t)e^{-st}dt + \int_1^2 \textsf{sqw}(t)e^{-st}dt && \text{Apply } \textstyle\int_a^b = \int_a^c + \int_c^b. \\
&= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt && \text{Use } \textsf{sqw}(x) = 1 \text{ on } 0 \le x < 1 \text{ and } \textsf{sqw}(x) = -1 \text{ on } 1 \le x < 2. \\
&= \frac{1}{s}(1 - e^{-s}) + \frac{1}{s}(e^{-2s} - e^{-s}) && \text{Evaluate each integral.} \\
&= \frac{1}{s}(1 - e^{-s})^2 && \text{Collect terms.}
\end{aligned}$$

An intermediate step is to compute the Laplace integral of $\textsf{sqw}(t)$:

$$\begin{aligned}
\mathcal{L}(\textsf{sqw}(t)) &= \frac{\int_0^2 \textsf{sqw}(t)e^{-st}dt}{1 - e^{-2s}} && \text{Periodic function formula, page 613.} \\
&= \frac{1}{s}(1 - e^{-s})^2 \frac{1}{1 - e^{-2s}}. && \text{Use the computation of } \mathcal{P} \text{ above.} \\
&= \frac{1}{s}\frac{1 - e^{-s}}{1 + e^{-s}}. && \text{Factor } 1 - e^{-2s} = (1 - e^{-s})(1 + e^{-s}). \\
&= \frac{1}{s}\frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}. && \text{Multiply the fraction by } e^{s/2}/e^{s/2}. \\
&= \frac{1}{s}\frac{\sinh(s/2)}{\cosh(s/2)}. && \text{Use } \sinh u = (e^u - e^{-u})/2, \ \cosh u = (e^u + e^{-u})/2. \\
&= \frac{1}{s}\tanh(s/2). && \text{Use } \tanh u = \sinh u / \cosh u.
\end{aligned}$$

To complete the computation of $\mathcal{L}(\textsf{sqw}(t/a))$, a change of variables is made:

$$\begin{aligned}
\mathcal{L}(\textsf{sqw}(t/a)) &= \int_0^\infty \textsf{sqw}(t/a)e^{-st}dt && \text{Direct transform.} \\
&= \int_0^\infty \textsf{sqw}(r)e^{-asr}(a)dr && \text{Change variables } r = t/a. \\
&= \frac{a}{as}\tanh(as/2) && \text{See } \mathcal{L}(\textsf{sqw}(t)) \text{ above.} \\
&= \frac{1}{s}\tanh(as/2)
\end{aligned}$$

- **Proof of $\mathcal{L}(a\,\textsf{trw}(t/a)) = \dfrac{1}{s^2}\tanh(as/2)$:**

The triangular wave is defined by $\textsf{trw}(t) = \int_0^t \textsf{sqw}(x)dx$.

$$\begin{aligned}
\mathcal{L}(a\,\textsf{trw}(t/a)) &= \frac{1}{s}(f(0) + \mathcal{L}(f'(t)) && \text{Let } f(t) = a\,\textsf{trw}(t/a). \text{ Use } \mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0), \text{ page 581.} \\
&= \frac{1}{s}\mathcal{L}(\textsf{sqw}(t/a)) && \text{Use } f(0) = 0, \ (a\textstyle\int_0^{t/a} \textsf{sqw}(x)dx)' = \textsf{sqw}(t/a). \\
&= \frac{1}{s^2}\tanh(as/2) && \text{Table entry for } \textsf{sqw}.
\end{aligned}$$

- **Proof of $\mathcal{L}(t^\alpha) = \dfrac{\Gamma(1+\alpha)}{s^{1+\alpha}}$:**

$$\begin{aligned}
\mathcal{L}(t^\alpha) &= \int_0^\infty t^\alpha e^{-st}dt && \text{Direct Laplace transform.} \\
&= \int_0^\infty (u/s)^\alpha e^{-u}du/s && \text{Change variables } u = st, \ du = sdt.
\end{aligned}$$

$$= \frac{1}{s^{1+\alpha}} \int_0^\infty u^\alpha e^{-u} du$$

$$= \frac{1}{s^{1+\alpha}} \Gamma(1+\alpha).$$

Where $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$, by definition.

The *generalized factorial function* $\Gamma(x)$ is defined for $x > 0$ and it agrees with the classical factorial $n! = (1)(2)\cdots(n)$ in case $x = n+1$ is an integer. In literature, $\alpha!$ means $\Gamma(1+\alpha)$. For more details about the Gamma function, see Abramowitz and Stegun [**?**], or `maple` documentation.

- **Proof of $\mathcal{L}(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$:**

$$\mathcal{L}(t^{-1/2}) = \frac{\Gamma(1+(-1/2))}{s^{1-1/2}}$$

Apply the previous formula.

$$= \frac{\sqrt{\pi}}{\sqrt{s}}$$

Use $\Gamma(1/2) = \sqrt{\pi}$.