Name KEY

2280 Final Exam S2018

Final Exam Differential Equations 2280

Monday, 30 April 2018, 12:45pm-3:15pm

Instructions: Time limit 120 minutes. No calculators, notes, tables or books. No answer check is expected. A correct answer without details counts 25%.

Chapters 1 and 2: Linear First Order Differential Equations

A **(a)** [40%] Solve $7\frac{d}{dt}v(t) = 11 + \frac{6}{t+1}v(t)$, $v(0) = 11$. Show all integrating factor steps.

A **(b)** [30%] Solve the linear homogeneous equation $x^2\frac{dy}{dx} = y + xy$.

A **(c)** [30%] The problem $x^2\frac{dy}{dx} = y + 2x + 2 + xy$ is both linear and separable. It can be solved by linear theory using superposition $y = y_h + y_p$, where y_p is an equilibrium solution. Find y_h and y_p.

a) $7v' - \frac{6}{t+1}v = 11$, $V_0 = 11$

$v' - \frac{6}{7(t+1)}v = \frac{11}{7} \Rightarrow w = e^{-\int \frac{6}{7}\cdot\frac{1}{t+1}} = e^{-\frac{6}{7}\ln|t+1|} = (t+1)^{-6/7}$

$(wv)' = \frac{11}{7}(t+1)^{-6/7}$

$wv = 11(t+1)^{1/7} + C$

$v = 11(t+1) + C$

$11 = 11 + C \Rightarrow C = 0$

$v = 11t + 11$

b) $x^2 y' - (1+x)y = 0$

$y' - \frac{1+x}{x^2}y = 0 \Rightarrow w = e^{-\int \frac{1+x}{x^2}} = e^{-\int \frac{1}{x^2} + \frac{1}{x}} = e^{\frac{1}{x} - \ln|x|}$

$y = \frac{c}{w} = Ce^{\ln|x| - \frac{1}{x}} = Ce^{-1/x}\cdot x$

c) $y' - \frac{1+x}{x^2}y = 0 \leadsto y_h = Ce^{\ln|x| - \frac{1}{x}} = Cxe^{-1/x}$

$0 = y + 2x + 2 + xy$

$0 = y(x+1) + 2(x+1)$

$y_p = \frac{-2(x+1)}{x+1} = -2$

$y = Cxe^{-1/x} - 2$

Chapter 3: Linear Equations of Higher Order

A **(a)** [10%] Solve for the general solution: $y'' - 6y' + 25y = 0$

A **(b)** [20%] Solve for the general solution: $y^{(5)} - 6y^{(4)} + 25y^{(3)} = 0$

A **(c)** [20%] An nth order linear homogeneous differential equation has characteristic equation $r(r^3 - r)^2(r^2 - 6r + 25)^2 = 0$. Solve for the general solution.

A **(d)** [20%] Construct the characteristic equation of a linear nth order homogeneous differential equation of least order n which has a particular solution $y(x) = x\cos(2x) + 3x^4e^x + e^x\sin(3x)$. excused

A **(e)** [30%] An nth order non-homogeneous differential equation is specified by its characteristic equation $r^2(r+1)^3(r^2+16) = 0$ and the forcing term $f(x) = x^2 + x^3e^{-x} + xe^{3x} + \sin(4x)$. Find the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate undetermined coefficients.

a) $y'' - 6y' + 25y = 0$

$r^2 - 6r + 25 = 0$

$(r-3)^2 + 16 = 0$

$r = 3 \pm 4i$

$\boxed{y = C_1e^{3t}\cos(4t) + C_2e^{3t}\sin(4t)}$

b) $y''''' - 6y'''' + 25y''' = 0$

$r^5 - 6r^4 + 25r^3 = 0$

$r^3(r^2 - 6r + 25) = 0$

$r = 0, 0, 0, 3 \pm 4i$

$y = C_1 + C_2t + C_3t^2 + C_4e^{3t}\cos(4t) + C_5e^{3t}\cos(4t)$

c) $r^3(r^2-1)^2(r^2-6r+25)^2 = 0$

$= 0, 0, 0, \pm 1, \pm 1, 3 \pm 4i$

$= C_1 + C_2t + C_3t^2 + C_4e^t + C_5e^{-t} + C_6te^t + C_7te^{-t} + C_8e^{3t}\cos(4t) + C_9e^{3t}\sin(4$

) $y(x) = x\cos(2x) + 3x^4e^x + e^x\sin(3x)$

$= \pm 2i, \pm 2i, 1, 1, 1, 1, 1, 1 \pm 3i$

$(r+4)^{4}(r-1)^5((r-1)^2+9) = 0$

e) Rule I:

Atoms: $1, x, x^2, e^{-x}, xe^{-x}, x^2e^{-x}, x^3e^{-x}, e^{3x}, xe^{3x}, \cos(4x), \sin(4x)$

New atoms: $x^2, x^3, x^4, x^3e^{-x}, x^4e^{-x}, x^5e^{-x}, x^6e^{-x}, e^{3x}, xe^{3x}, x\cos(4x), x\sin(4x)$

$y_p =$ l.c. of

ule #2:

$r^2(r+1)^3(r^2+16) = 0$

$= 0, 0, -1, -1, -1, \pm 4i$

toms: $1, x, e^{-x}, xe^{-x}, x^2e^{-x}, \cos(4x), \sin(4x)$

Chapters 4 and 5: Systems of Differential Equations

A **(a)** [10%] Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 4 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Find all eigenpairs of A and then write the solution of $\vec{\mathbf{x}}(t)$ of $\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t)$ according to the Eigenanalysis Method.

A **(b)** [20%] Find the general solution of the 2×2 system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

according to the Cayley-Hamilton-Ziebur Method, using the textbook's Chapter 4 shortcut.

A **(c)** [10%] Assume a 3×3 system $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}}$ has a scalar general solution

$$u_1(t) = -c_1e^{4t} + c_2e^{8t}, \quad u_2(t) = c_1e^{4t} + c_2e^{8t}, \quad u_3(t) = c_3e^{t}$$

Compute a 3×3 fundamental matrix $\Phi(t)$ and then write a formula for the exponential matrix e^{At}. Do not simplify any formula.

A **(d)** [20%] Consider the 3×3 linear homogeneous system

$$\begin{cases} x' &= 5x - y \\ y' &= -x + 5y, \\ z' &= x + z \end{cases} \quad \text{or} \quad \frac{d}{dt}\vec{\mathbf{u}}(t) = \begin{pmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}\vec{\mathbf{u}}(t).$$

Solve the system by the most efficient method.

a) $A = \begin{pmatrix} 0 & 1 & 1 \\ 4 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix}$, $|A - \lambda I| = (3-\lambda)(\lambda^2 - 4) = 0$

$\lambda = 3, 2, -2$

$A - 2I = \begin{pmatrix} -2 & 1 & 1 \\ 4 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$A + 2I = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix} \cdot \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$

$A - 3I = \begin{pmatrix} -3 & 1 & 1 \\ 4 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix}$

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + c_3 e^{3t}\begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix}$$

b) $\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} = A$, $\begin{cases} x' = 6x + 2y \\ y' = 2x + 6y \end{cases}$

$|A - \lambda I| = (6-\lambda)(6-\lambda) - 4 = 0$

$= \lambda^2 - 12\lambda + 32$

$= (\lambda - 8)(\lambda - 4)$

$\lambda = 8, 4$

$y = \frac{x'}{2} - 3x$

$y = 2c_1 e^{4t} + 4c_2 e^{8t} - 3c_1 e^{4t} - 3c_2 e$

$$\boxed{\begin{aligned} x &= c_1 e^{4t} + c_2 e^{8t} \\ y &= -c_1 e^{4t} + c_2 e^{8t} \end{aligned}}$$

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c) $\begin{cases} u_1 = -c_1 e^{4t} + c_2 e^{8t} \\ u_2 = c_1 e^{4t} + c_2 e^{8t} \\ u_3 = c_3 e^{t} \end{cases} \Rightarrow \Phi = \begin{pmatrix} -e^{4t} & e^{8t} & 0 \\ e^{4t} & e^{8t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}$

$\Phi(0) = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\left(\begin{array}{ccc:ccc} -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ so $\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$e^{At} = \Phi(t)\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} e^{4t} + e^{8t} & -e^{4t} + e^{8t} & 0 \\ -e^{4t} + e^{8t} & e^{4t} + e^{8t} & 0 \\ 0 & 0 & 2e^{t} \end{pmatrix}$

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d) $\begin{cases} x' = 5x - y \\ y' = -x + 5y \\ z' = x + z \end{cases} \Rightarrow A = \begin{pmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $|A - \lambda I| = (1-\lambda)\left((5-\lambda)^2 - 1\right) = 0$

$= (1-\lambda)(\lambda^2 - 10\lambda + 24)$

$= (1-\lambda)(\lambda - 6)(\lambda - 4)$

$\lambda = 1, 4, 6$

$A - I = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$A - 4I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix} \cdot \vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

$$A - 6I = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -5 \end{pmatrix} \cdot \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$$

$$\vec{u} = c_1 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + c_3 e^{6t} \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$$

$$\begin{cases} x = 3c_2 e^{4t} + 5c_3 e^{6t} \\ y = 3c_2 e^{4t} - 5c_3 e^{6t} \\ z = c_1 e^{t} + c_2 e^{4t} + c_3 e^{6t} \end{cases}$$

Chapter 6: Dynamical Systems

Consider the nonlinear dynamical system

$$\begin{cases} x' = 16x - 2x^2 - xy, \\ y' = 14y - 2y^2 - xy \end{cases} \quad (1)$$

A **(a)** [20%] Find the equilibrium points for nonlinear system (1).

A **(b)** [20%] Compute the Jacobian matrix $J(x, y)$ for nonlinear system (1). Then evaluate $J(x, y)$ at each of the equilibrium points found in part (a).

A **(c)** [30%] Consider nonlinear system (1). Classify the linearization at each equilibrium point found in part (a) as a node, spiral, center, saddle. Do not sub-classify a node.

A **(d)** [30%] Consider nonlinear system (1). Determine the possible classifications of node, spiral, center or saddle and corresponding stability for each equilibrium determined in part (a), according to the **Pasting Theorem**, which is Theorem 2 in section 6.2 (Stability of Almost Linear Systems).

a) $0 = x(16-2x-y)$

$0 = y(14-2y-x)$

$(0,0),\ (8,0),\ (0,7),\ (6,4)$

$16-2x=0 \Rightarrow x=8$

$14-2y=0 \Rightarrow y=7$

$y = 16-2x$

$14 = 2y+x$

$14 = 32-4x+x$

$-18 = -3x \Rightarrow x=6$ and $y=4$

b) $J(x,y) = \begin{pmatrix} 16-4x-y & -x \\ -y & 14-4y-x \end{pmatrix}$

$J(0,0) = \begin{pmatrix} 16 & 0 \\ 0 & 14 \end{pmatrix} = A_1$

$J(8,0) = \begin{pmatrix} -16 & -8 \\ 0 & 6 \end{pmatrix} = A_2$

$J(6,4) = \begin{pmatrix} -12 & -6 \\ -4 & -8 \end{pmatrix} = A_4$

$J(0,7) = \begin{pmatrix} 9 & 0 \\ -7 & -14 \end{pmatrix} = A_3$

c). $|A_1 - \lambda I| = (16-\lambda)(14-\lambda) = 0$

$\lambda = 16, 14$

Atoms: e^{16t}, e^{14t}

unstable node at (0,0)

$|A_2 - \lambda I| = (-16-\lambda)(6-\lambda) = 0$

$\lambda = -16, 6$

Atoms: e^{-16t}, e^{6t}

unstable saddle at (8,0)

$A_3 - \lambda I| = (9-\lambda)(-14-\lambda) = 0$

$\lambda = 9, -14$

Atoms: e^{9t}, e^{-14t}

unstable saddle at (0,7)

$A_4 - \lambda I| = (-12-\lambda)(-8-\lambda) - 24 = 0$

$= \lambda^2 - 20\lambda + 96 - 24$

$= (\lambda - 10)^2 - 28$

$\lambda = 10 \pm \sqrt{28}$

Atoms: $\sim e^{t}, \sim e^{t}$

unstable node at (6,4)

d) All of them paste (no equal eigenvalues or purely imaginary eigenvalues) and stability transfers from linear case

(0,0): unstable node

(8,0): unstable saddle

(0,7): unstable saddle

(6,4): unstable node

Chapter 7: Laplace Theory

A **(a)** [20%] Solve for $f(t)$ in the relation $\mathcal{L}(f) = \left(\frac{d}{ds}\mathcal{L}\left(t^2 e^{5t}\sin t\right)\right)\Big|_{s\to s+2}$.

A **(b)** [20%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^t + t\,e^{-t}\sin(2t)$. excused

A **(c)** [30%] Consider the forced linear dynamical system

$$\begin{cases} x' &= 5x - y + 2t, \\ y' &= -x + 5y + 1. \end{cases}$$

Show that subject to initial states $x(0) = 0$, $y(0) = 0$ the solution $x(t), y(t)$ satisfies

$$\mathcal{L}(x(t)) = \frac{s-10}{s^2(s-4)(s-6)}, \quad \mathcal{L}(y(t)) = \frac{s^2-5s-2}{s^2(s-4)(s-6)}.$$

A **(d)** [30%] Solve for $x(t)$ in the relation $\mathcal{L}(x(t)) = \frac{s-10}{s^2(s-4)(s-6)}$. Leave the partial fraction constants unevaluated.

a) $\mathcal{L}(f) = \mathcal{L}\left((-t)t^2e^{5t}\sin t\right)\big|_{s\to s+2}$

$f(t) = -t^3e^{3t}\sin t$

b) $f(t) = -te^t + te^{-t}\sin(2t)$

$\mathcal{L}(f) = \frac{d}{ds}\left(\frac{1}{s-1}\right) - \frac{d}{ds}\left(\frac{2}{(s+1)^2+4}\right)$ ✓

$\mathcal{L}(f) = \frac{-1}{(s-1)^2} - 2\left(\frac{-2(2s+2)}{(s^2+2s+5)^2}\right)$

$= \frac{-1}{(s-1)^2} + 8\cdot\frac{s+1}{(s^2+2s+5)^2}$ ok

c) $\begin{cases} x' = 5x - y + 2t \\ y' = -x + 5y + 1 \end{cases}$

$s\mathcal{L}(x) - \cancel{x(0)}^{0} = 5\mathcal{L}(x) - \mathcal{L}(y) + 2\mathcal{L}(t) \Rightarrow \frac{2}{s^2} = (s-5)\mathcal{L}(x) + \mathcal{L}(y)$

$s\mathcal{L}(y) - \cancel{y(0)}^{0} = -\mathcal{L}(x) + 5\mathcal{L}(y) + \mathcal{L}(1) \Rightarrow \frac{1}{s} = \mathcal{L}(x) + (s-5)\mathcal{L}(y)$

$\Delta = \begin{vmatrix} s-5 & 1 \\ 1 & s-5 \end{vmatrix} = (s-5)^2 - 1$

$\Delta_1 = \begin{vmatrix} 2/s^2 & 1 \\ 1/s & s-5 \end{vmatrix} = \frac{2(s-5)-s}{s^2}$

$\Delta_2 = \begin{vmatrix} s-5 & 2/s^2 \\ 1 & 1/s \end{vmatrix} = \frac{s(s-5)-2}{s^2}$

$\mathcal{L}(x) = \frac{2s-10-s}{s^2(s^2-10s+24)} = \frac{s-10}{s^2(s-4)(s-6)}$

$\mathcal{L}(y) = \frac{s^2-5s-2}{s^2(s^2-10s+24)} = \frac{s^2-5s-2}{s^2(s-4)(s-6)}$

d). $\mathcal{L}(x) = \dfrac{s-10}{s^2(s-4)(s-6)}$

$$\frac{s-10}{s^2(s-4)(s-6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-4} + \frac{D}{s-6}$$

So, $\mathcal{L}(x) = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-4} + \dfrac{D}{s-6}$

$$x(t) = A + Bt + Ce^{4t} + De^{6t}$$

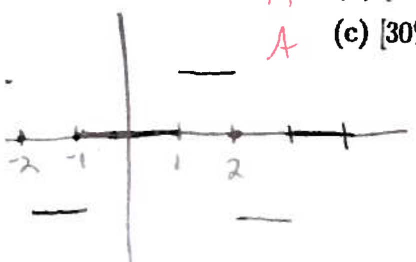
Chapter 9: Fourier Series and Partial Differential Equations

In parts (a) and (b), let $f_0(x) = -1$ on the interval $-2 < x < -1$, $f_0(x) = 1$ on the interval $1 < x < 2$, $f_0(x) = 0$ for all other values of x on $-2 \le x \le 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [20%] Compute the Fourier coefficients of $f(x)$ on $[-2, 2]$.

(b) [10%] Find all values of x in $|x| < 4$ which will exhibit Gibb's over-shoot.

(c) [30%] **Heat Conduction in a Rod.** Solve the rod problem on $0 \le x \le 2$, $t \ge 0$:



odd

$L = 2$

$$\begin{cases} u_t &= u_{xx}, \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= 2\sin(\pi x) + 5\sin(2\pi x) \end{cases}$$

(d) [40%] **Vibration of a Finite String.** Solve the finite string vibration problem on $0 \le x \le \cancel{4}\ 5$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &= 16u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(5,t) &= 0, \\ u(x,0) &= \sin(5\pi x) + 2\sin(7\pi x), \\ u_t(x,0) &= \sin(7\pi x) + \sin(10\pi x). \end{cases}$$

a) $a_n = \int_{-2}^{2} \underbrace{\underbrace{f_0(x)}_{\text{odd}} \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}}_{\text{odd}} dx = 0 \quad \forall\, n \ge 0$

$b_n = \frac{2}{L}\int_0^2 f_0(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^1 0\,dx + 2\int_1^2 \sin\left(\frac{n\pi x}{L}\right)dx$

$= \frac{2L}{n\pi \cdot L} \cdot -\cos\left(\frac{n\pi x}{L}\right)\Big|_{x=1}^{x=2}$

$= \frac{4}{n\pi}\left(-\cos(n\pi) + \cos\left(\cancel{\frac{n\pi}{2}}\right)^{0}\right)$ error

zero → only n odd

$= \frac{-4}{n\pi}\cos(n\pi) \quad \forall\, n \ge 1$

$= \frac{2}{n\pi}\left(-\cos(n\pi) + \cos\left(\frac{n\pi}{2}\right)\right)$

b) $x = -3, -2, -1, 1, 2, 3$

c) $k = 1$, $L = 2$

$X(x) = \sin\left(\frac{n\pi x}{L}\right)$, $T(t) = e^{-\left(\frac{n\pi}{L}\right)^2 \cdot kt}$

$c_n = 0 \ \forall\, n$ except $c_2 = 2$ and $c_4 = 5$

$u(x,t) = 2\sin(\pi x)e^{-\pi^2 t} + 5\sin(2\pi x)^{-4\pi^2 t}$

d) $c=4,\ L=5$

$u(x,t) = X(x)T(t)$, $X(x) = \sin\left(\frac{n\pi x}{5}\right)$, $T(t) = c_n\cos\left(\frac{4n\pi t}{5}\right) + d_n \sin\left(\frac{4n}{5}\right.$

$u(x,0) = \sum_{n=1}^{\infty} X(x)T(0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{5}\right) = \sin(5\pi x) + 2\sin(7\pi x)$

$c_n = 0 \ \forall n$ except $c_{25} = 1,\ c_{35} = 2$

$u_t(x,0) = \sum_{n=1}^{\infty} X(x)T'(0) = \sum_{n=1}^{\infty} \frac{4d_n \cdot n\pi}{5} \sin\left(\frac{n\pi x}{5}\right) = \sin(7\pi x) + \sin(10\pi x)$

$n=35:\ 28\pi d_{35} = 1 \Rightarrow d_{35} = \frac{1}{28\pi}$

$n=50:\ 40\pi d_{50} = 1 \Rightarrow d_{50} = \frac{1}{40\pi}$

$d_n = 0$ except $d_{35} = \frac{1}{28\pi}$ and $d_{50} = \frac{1}{40\pi}$

Then, $u(x,t) = \sin(5\pi x)\cos(20\pi t) + 2\sin(7\pi x)\cos(28\pi t)$
$+ \frac{1}{28\pi}\sin(7\pi x)\sin(28\pi t) + \frac{1}{40\pi}\sin(10\pi x)\sin(40\pi t)$