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Differential Equations 2280

Midterm Exam 3

Exam Date: 13 April 2018 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Chapter 3 – Nth Order Differential Equations

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Problem (1a) [40%] Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 289x = 280\cos(3t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time, please don't convert your answer.**

$r^2 + 289 = 0 \Rightarrow r = \pm\sqrt{289}\,i = \pm 17i$

x_p: rule I atoms: $\cos(3t), \sin(3t)$; no rule 2 conflict

$x_p = c_1\cos(3t) + c_2\sin(3t)$

$x' = -3c_1\sin(3t) + 3c_2\cos(3t)$

$x'' = -9c_1\cos(3t) - 9c_2\sin(3t)$

$-9(c_1\cos(3t) + c_2\sin(3t)) + 289(c_1\cos(3t) + c_2\sin(3t)) = 280\cos($

$-9c_1 + 289c_1 = 280 \Rightarrow c_1 = 1$

$-9c_2 + 289c_2 = 0 \Rightarrow c_2 = 0$

$x_p = \cos(3t)$

$x = x_h + x_p \quad x_h = d_1\cos(17t) + d_2\sin(17t)$

$x = d_1\cos(17t) + d_2\sin(17t) + \cos(3t) \quad x(0) = 0 \Rightarrow d_1 = -1$

$x' = -17d_1\sin(17t) + 17d_2\cos(17t) - 3\sin(3t) \quad x'(0) = 0 \Rightarrow d_2 = 0$

$x = \cos(3t) - \cos(17t)$

Chapter 3 – Nth Order Differential Equations

Problem (1b) [30%] Let $f(x) = x^2 e^{-x}\sin(x) - x e^x + x^5\cos(2x)$. Find the characteristic equation of a linear homogeneous scalar differential equation of least order such that $y = f(x)$ is a solution. Kindly leave the characteristic equation in factored form, unexpanded.

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$x^2e^{-x}\sin(x)$: $e^{-x}\cos(x), e^{-x}\sin(x), xe^{-x}\cos(x), xe^{-x}\sin(x), x^2e^{-x}\cos(x), x^2e^{-x}\sin(x)$

$r = -1 \pm i$ mult. 3

xe^x : e^x, xe^x

$r = 1, 1$

$x^5\cos(2x)$: $\cos(2x), \sin(2x), \ldots, x^5\cos(2x), x^5\sin(2x)$

$r = \pm 2i$, mult. 6

char. equation: $((r+1)^2+1)^3 (r-1)^2 (r^2+2)^6 = 0$

Chapter 3 – Nth Order Differential Equations

Problem (1c) [40%] Consider the mechanical oscillation equation (M) and electrical current equation (E):

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$$\begin{cases} \textbf{(M)} \quad 2x'' + 5x' + 50x & = \quad 100\cos(\omega t), \\ \textbf{(E)} \quad 2I'' + 5I' + 50I & = \quad 100\,\omega\cos(\omega t). \end{cases}$$

Part 1. Explain in terms of of the three coefficients 2, 5, 50 why the mechanical equation must be overdamped. Then give an example of three positive coefficients for which the mechanical equation is critically damped.

$b^2 - 4ac = 25 - 400 = -375 \Rightarrow$ complex roots

✓ critically damped $\Rightarrow r_1, r_2$ roots of char. eq. $r_1 = r_2$
$m = 1,\ k = 2,\ c = 2$

Part 2. Determine the practical resonance frequency ω for each equation.

$$\omega_M = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{\frac{50}{2} - \frac{25}{8}} = \sqrt{\frac{200-25}{8}} = \sqrt{\frac{175}{8}}$$

$$\omega_I = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{2}{50}}} = \sqrt{25} = 5$$

Chapters 4 and 5 – Systems of Differential Equations

Theorem. (Eigenanalysis Method) If A is a real 3×3 matrix with eigenpairs $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$, $(\lambda_3, \vec{v}_3)$, then the system $\vec{x}' = A\vec{x}$ has general solution

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + c_3\vec{v}_3 e^{\lambda_3 t}.$$

Theorem. (Cayley-Hamilton-Ziebur). The components of solution $\vec{x}$ of $\vec{x}'(t) = A\vec{x}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$.

Definition. Let A be an $n \times n$ real matrix. An augmented matrix $\Phi(t)$ of n independent solutions of $\vec{x}'(t) = A\vec{x}(t)$ is called a **fundamental matrix**. It is known that the general solution is $\vec{x}(t) = \Phi(t)\vec{c}$, where $\vec{c}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, $|\Phi(0)| \neq 0$.

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Problem (2a) [30%] Assume a 3×3 matrix A has eigenvalues $\lambda = 3, 5 \pm 4i$. Display a solution formula for the vector solution $\vec{u}(t)$ to system $\frac{d}{dt}\vec{u} = A\vec{u}$, inserting what is known what is known from the eigenvalue information (supplied above). **To save time**, observe that it is impossible to solve for coefficients, because A is not known, only its eigenvalues.

$$\vec{u}(t) = C_1 e^{3t}\vec{V}_1 + C_2 e^{5t}\cos(4t)\vec{V}_2 + C_3 e^{5t}\sin(4t)\vec{V}_3$$

$\vec{V}_1$ is the eigenvector associated w/ $\lambda = 3$

$\vec{V}_2, \vec{V}_3$ are eigenvectors associated w/ $\lambda = 5 \pm 4i$

Problem (2b) [40%] A linear cascade satisfies $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where the 4×4 triangular matrix and vector $\vec{x}$ are defined by

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

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Use any method to find the vector general solution $\vec{x}(t)$ of $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

$x_4' = 3x_4 \quad x_4 = \frac{c_1}{e^{-3t}} = c_1 e^{3t}$

$x_3' = 2x_3 + x_4 \quad x_3' - 2x_3 = c_1 e^{3t} \quad w = e^{-2t}$

$\frac{(w x_3)'}{w} = c_1 e^{3t} \quad (e^{-2t} x_3)' = c_1 e^{t}$ integrate both sides

$e^{-2t} x_3 = c_1 e^{t} + c_2$

$x_3 = c_1 e^{3t} + c_2 e^{2t}$

$x_2' = 2x_2 \quad x_2 = \frac{c_3}{e^{-2t}} = c_3 e^{2t}$

$x_1' = 2x_1 + x_2 \quad x_1' - 2x_1 = c_3 e^{2t} \quad v = e^{-2t}$

$\frac{(e^{-2t} x_1)'}{e^{-2t}} = c_3 e^{2t} \quad (e^{-2t} x_1)' = c_3$ integrate both sides

$e^{-2t} x_1 = c_3 t + c_4$

$x_1 = c_3 t e^{2t} + c_4 e^{2t}$

$$\boxed{\begin{aligned} x_1 &= c_3 t e^{2t} + c_4 e^{2t} \\ x_2 &= c_3 e^{2t} \\ x_3 &= c_1 e^{3t} + c_2 e^{2t} \\ x_4 &= c_1 e^{3t} \end{aligned}}$$

Chapters 4 and 5 – Systems of Differential Equations

Problem (2c) [40%] A linear cascade satisfies $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where the 4×4 triangular matrix A, vector solution $\vec{x}(t)$ and initial condition $\vec{x}(0)$ are defined by

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$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Apply Laplace's method to obtain a 4×4 system for $\mathcal{L}(x_1), \mathcal{L}(x_2), \mathcal{L}(x_3), \mathcal{L}(x_4)$. **To save time,** don't use Cramer's Rule to find $\mathcal{L}(x_1), \mathcal{L}(x_2), \mathcal{L}(x_3), \mathcal{L}(x_4)$. Don't use Laplace tables and don't find the solution $\vec{x}(t)$! Your solution can use scalar equations or the vector-matrix equation $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

$$\begin{aligned}
x_1' &= 2x_1 + x_2 \quad \Rightarrow \quad & x_1' - 2x_1 - x_2 &= 0 \\
x_2' &= 2x_2 & x_2' - 2x_2 &= 0 \\
x_3' &= 2x_3 + x_4 & x_3' - 2x_3 - x_4 &= 0 \\
x_4' &= 3x_4 & x_4' - 3x_4 &= 0
\end{aligned}$$

take $\mathcal{L}(RHS) = \mathcal{L}(LHS)$

not needed, but ok.

$$s\mathcal{L}(x_1) - \overset{1}{x_1(0)} - 2\mathcal{L}(x_1) - \mathcal{L}(x_2) = \mathcal{L}(0)$$

$$s\mathcal{L}(x_2) - \overset{0}{x_2(0)} - 2\mathcal{L}(x_2) = \mathcal{L}(0)$$

$$s\mathcal{L}(x_3) - \overset{0}{x_3(0)} - 2\mathcal{L}(x_3) - \mathcal{L}(x_4) = \mathcal{L}(0)$$

$$s\mathcal{L}(x_4) - \overset{1}{x_4(0)} - 3\mathcal{L}(x_4) = \mathcal{L}(0)$$

ok

$$\begin{bmatrix} s-2 & -1 & 0 & 0 \\ 0 & s-2 & 0 & 0 \\ 0 & 0 & s-2 & -1 \\ 0 & 0 & 0 & s-3 \end{bmatrix} \begin{bmatrix} \mathcal{L}(x_1) \\ \mathcal{L}(x_2) \\ \mathcal{L}(x_3) \\ \mathcal{L}(x_4) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

ok

Chapters 4 and 5 – Systems of Differential Equations

Problem (2d) [30%] The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = 2x + 7y, \quad y' = -7x + 2y,$$

which has complex eigenvalues $\lambda = 2 \pm 7i$.

> **Part 1.** Show the details of the method, finally displaying formulas for $x(t), y(t)$.
> **Part 2.** Report a fundamental matrix $\Phi(t)$.
> **Part 3.** Use **Part 2** to find the exponential matrix e^{At}.

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(1) $x' - 2x = 7y$

$\lambda = 2 \pm 7i \Rightarrow$ atoms: $e^{2t}\cos(7t),\ e^{2t}\sin(7t)$

$x = c_1 e^{2t}\cos(7t) + c_2 e^{2t}\sin(7t)$

$x' = 2c_1 e^{2t}\cos(7t) - 7c_1 e^{2t}\sin(7t) + 2c_2 e^{2t}\sin(7t) + 7c_2 e^{2t}\cos(7t)$

$x' - 2x = (2c_1 + 7c_2)e^{2t}\cos(7t) + (2c_2 - 7c_1)e^{2t}\sin(7t) - 2(c_1 e^{2t}\cos(7t) + c_2 e^{2t}\sin(7t)$

$= 7c_2 e^{2t}\cos(7t) - 7c_1 e^{2t}\sin(7t)$

$y = \frac{LHS}{7} = c_2 e^{2t}\cos(7t) - c_1 e^{2t}\sin(7t)$, $x = c_1 e^{2t}\cos(7t) + c_2 e^{2t}\sin(7t)$

(2) $\Phi(t) = \begin{bmatrix} e^{2t}\cos(7t) & e^{2t}\sin(7t) \\ -e^{2t}\sin(7t) & e^{2t}\cos(7t) \end{bmatrix}$ $\quad \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(3) $e^{At} = \Phi(t)\Phi^{-1}(0) = \Phi(t) I = \Phi(t)$

Chapter 6, Linear and Nonlinear Dynamical Systems

Problem (3a) [20%] Classify the unique equilibrium $\vec{u} = \vec{0}$ as a saddle, center, spiral or node and report if it is stable or unstable. Sub-classification of a node into improper or proper node is not required.

$$A \qquad \frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} \vec{u}$$

$|A - \lambda I| = 0 \quad (1-\lambda)^2 - 16 = 0 \quad \lambda^2 - 2\lambda + 1 - 16 = 0$

$\lambda^2 - 2\lambda - 15 = 0 \quad (\lambda - 5)(\lambda + 3) = 0 \quad \lambda = 5, -3$

atoms : $\underset{a_1}{e^{5t}}, \underset{a_2}{e^{-3t}}$

no trig functions in atoms $\Rightarrow$ non-rotating

$\lim_{t \to \infty} a_1 = \infty$

$\lim_{t \to \infty} a_2 = 0$

This is a saddle and is unstable

Chapter 6, Linear and Nonlinear Dynamical Systems

Problem (3b) [30%] Consider the nonlinear dynamical system

$$\begin{cases} x' &= e^x + 2y - 5, \\ y' &= -4x + e^{y-2} - 1 \end{cases}$$

$e^y e^{-2}$

An equilibrium point is $x = 0$, $y = 2$. Compute the Jacobian matrix $J(x, y)$ of the linearized system at this equilibrium point.

$J(x,y) = \begin{bmatrix} e^x & 2 \\ -4 & e^{y-2} \end{bmatrix}$ $\quad J(0,2) = \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix}$

Chapter 6, Linear and Nonlinear Dynamical Systems

Problem (3c) [30%] Consider the nonlinear system $\begin{cases} x' &= e^x + 2y - 5, \\ y' &= -4x + e^{y-2} - 1 \end{cases}$

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(Part 1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$, where A is the Jacobian matrix of this system at $x = 0$, $y = 2$.

$$J(x,y) = \begin{bmatrix} e^x & 2 \\ -4 & e^{y-2} \end{bmatrix} \qquad J(0,2) = \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix}$$

$|A - \lambda I| = 0 \quad (1-\lambda)^2 + 8 = 0 \quad \lambda^2 - 2\lambda + 10 = 0 \quad \lambda = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$

$\lambda = 1 \pm \sqrt{8}\, i$

excused

atoms: $\underbrace{e^t \cos(3t)}_{a_1}, \underbrace{e^t \sin(3t)}_{a_2}$

trig functions w/ exponentials $\Rightarrow$ spiral

$\lim_{t \to \infty} a_1 = \infty$

$\lim_{t \to \infty} a_2 = \infty$

$\Rightarrow$ unstable

unstable, spiral

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(Part 2) Apply the Pasting Theorem, which is Theorem 2 in section 6.2, to classify $x = 0$, $y = 2$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

Paste Theorem exceptions: (1) equal real roots: node or spiral (stability pastes)

(2) purely imaginary roots: center or spiral

all other cases paste to non-linear system.

$x=0, y=2$ has complex roots with real parts $\neq 0$ so it pastes into the nonlinear system by the paste theorem.

unstable, spiral

Chapter 6, Linear and Nonlinear Dynamical Systems

Problem (3d) [20%] State the hypotheses and the conclusions of the *Pasting Theorem* used in problem **(3c)** above. Accuracy and completeness expected. Append to the statement the answer to this question:

A **For which figures, among spiral, center, saddle and node, is stability of the pasted figure onto the nonlinear phase portrait not determined?**

① for r_1, r_2 roots of the charateristic equation
if $r_1 = r_2$ the portrait will either be a node or a spiral
$r_1, r_2 < 0 \Rightarrow$ stable
$r_1, r_2 > 0 \Rightarrow$ unstable

② for purely imaginary roots the portrait in the non linear system will be either a center or a spiral. and either unstable or stable

③ for all other cases the portrait from the linear system passes to the non linear system