Differential Equations 2280

Midterm Exam 3

Exam Date: 14 April 2017 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Chapter 3

1. **(Linear Constant Equations of Order n)**

   (a) [30%] Find by variation of parameters a particular solution y_p for the equation $y'' = x + x^2$. Show all steps in variation of parameters. Check the answer by quadrature.

Chapter 3

(b) [40%] Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 256x = 247\cos(3t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time**, please don't convert your answer.

Chapter 3

(c) [30%] Let $f(x) = x^2 \cos(x) - x(e^x + 1)$. Find the characteristic equation of a linear homogeneous scalar differential equation of least order such that $y = f(x)$ is a solution.

Use this page to start your solution.

Chapters 4 and 5

2. **(Systems of Differential Equations)**

(a) [30%] Assume a 3×3 matrix A has eigenvalues $\lambda = 3, 4, 5$. State the Cayley-Hamilton-Ziebur theorem for this example. Then display a solution formula for the vector solution $\vec{u}(t)$ to system $\frac{d}{dt}\vec{u} = A\vec{u}$, inserting what is known what is known from the eigenvalue information (supplied above).

Chapters 4 and 5

(b) [40%] A linear cascade, typically found in brine tank models, satisfies $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where the 4×4 triangular matrix is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Part 1. Use the linear integrating factor method to find the vector general solution $\vec{x}(t)$ of $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

Chapters 4 and 5

(b) [40%] A linear cascade, typically found in brine tank models, satisfies $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ where the 4×4 triangular matrix is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Part 2. Laplace's method applies to this example. Explain in a paragraph of text how to apply Laplace's method to this 4×4 system. Don't use Laplace tables and don't find the solution! The explanation can use scalar equations or the vector-matrix equation $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

Chapters 4 and 5

Background for (c). Let A be an $n \times n$ real matrix. An augmented matrix $\Phi(t)$ of n independent solutions of $\vec{x}'(t) = A\vec{x}(t)$ is called a **fundamental matrix**. It is known that the general solution is $\vec{x}(t) = \Phi(t)\vec{c}$, where $\vec{c}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, $|\Phi(0)| \neq 0$.

(c) [30%] The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = x + 5y, \quad y' = -5x + y,$$

which has complex eigenvalues $\lambda = 1 \pm 5i$.

Part 1. Show the details of the method, finally displaying formulas for $x(t), y(t)$.
Part 2. Report a fundamental matrix $\Phi(t)$.
Part 3. Use **Part 2** to find the exponential matrix e^{At}.

Use this page to start your solution.

Chapter 6

3. (Linear and Nonlinear Dynamical Systems)

(a) [20%] Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node. Sub-classification into improper or proper node is not required.

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \vec{u}$$

(b) [30%] Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - 2y + 32, \\ y' &= 2x(x - 2y). \end{aligned}$$

An equilibrium point is $x = -8$, $y = -4$. Compute the Jacobian matrix of the linearized system at this equilibrium point.

(c) [30%] Consider the soft nonlinear spring system $\begin{cases} x' &= y, \\ y' &= -5x - 2y + \frac{5}{4}x^3. \end{cases}$

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$, where A is the Jacobian matrix of this system at $x = 2$, $y = 0$.

(2) Apply the Pasting Theorem to classify $x = 2$, $y = 0$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

(d) [20%] State the hypotheses and the conclusions of the *Pasting Theorem* used in part (c) above. Accuracy and completeness expected.

Use this page to start your solution.