Differential Equations 2280

Midterm Exam 3

Exam Date: 24 April 2015 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Chapter 3

1. **(Linear Constant Equations of Order n)**

(a) [30%] Find by variation of parameters a particular solution y_p for the equation $y'' = 2 + 6x$. Show all steps in variation of parameters. Check the answer by quadrature.

(b) [10%] A particular solution of the equation $LI'' + RI' + (1/C)I = I_0 \cos(10t)$ happens to be $I(t) = 5\cos(10t) + e^{-2t}\sin(\sqrt{17}t) - \sqrt{17}\sin(10t)$. Assume L, R, C all positive. Find the unique periodic steady-state solution I_{SS}.

(c) [40%] Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 39\cos(5t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time**, please don't convert to phase-amplitude form.

(d) [10%] Given $5x''(t) + 2x'(t) + 2x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = 2$, $k = 2$, determine if the equation is over-damped , critically damped or under-damped.
To save time, do not solve for $x(t)$.

(e) [10%] Determine the practical resonance frequency ω for the spring-mass equation

$$2x'' + 7x' + 50x = 500\cos(\omega t).$$

Answers and Solution Details:

All in progress.

Part (a) Answer: $y_p = x^2 + x^3$.

Variation of Parameters.

Solve $y'' = 0$ to get $y_h = c_1 y_1 + c_2 y_2$, $y_1 = 1$, $y_2 = x$. Compute the Wronskian $W = y_1 y_2' - y_1' y_2 = 1$. Then for $f(x) = 2 + 6x$,

$y_p = y_1 \int y_2 \frac{-f}{W} dx + y_2 \int y_1 \frac{f}{W} dx,$

$y_p = 1 \int -x(2+6x) dx + x \int 1(2+6x) dx,$

$y_p = -1(x^2 + 2x^3) + x(2x + 3x^2),$

$y_p = x^2 + x^3.$

This answer is checked by quadrature, applied twice to $y'' = 2 + 6x$ with initial conditions zero.

Part (b) It has to be the terms left over after striking out the transient terms, those terms with limit zero at infinity. Then $x_{\text{SS}}(t) = 5\cos(10t) - \sqrt{17}\sin(10t)$.

Part (c) The answer is $x(t) = -\cos(8t) + \cos(5t)$.

Use undetermined coefficients trial solution $x = d_1 \cos 5t + d_2 \sin 5t$. Then $d_1 = 1$, $d_2 = 0$, and finally $x_p(t) = \cos(5t)$. The characteristic equation $r^2 + 64 = 0$ has roots $\pm 8i$ with corresponding Euler solution atoms $\cos(8t), \sin(8t)$. Then $x_h(t) = c_1 \cos(8t) + c_2 \sin(8t)$. The general solution is $x = x_h + x_p$. Now use $x(0) = x'(0) = 0$ to determine $c_1 = -1, c_2 = 0$, which implies the particular solution $x(t) = -\cos(8t) + \cos(5t)$.

Part (d) Use the quadratic formula to decide. The number under the radical sign in the formula, called the discriminant, is $b^2 - 4ac = 2^2 - 4(5)(2) = -36$, therefore there are two complex conjugate roots and the equation is **under-damped**. Alternatively, factor $5r^2 + 2r + 2$ to obtain the roots and atoms, then classify as **under-damped**.

Part (e) $\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{\frac{151}{8}}$.

Use this page to start your solution.

Chapters 4 and 5

2. **(Systems of Differential Equations)**

(a) [30%] Display eigenanalysis details for the 3×3 matrix

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

then display the vector general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(b) [40%] The 3×3 triangular matrix

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

represents a linear cascade, such as found in brine tank models.

Part 1. Use the linear integrating factor method to find the vector general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

Part 2. Explain why the eigenanalysis method fails for this example.

(c) [30%] The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = 5x + 4y, \quad y' = -4x + 5y,$$

which has complex eigenvalues $\lambda = 5 \pm 4i$.

Part 1. Show the details of the method, finally displaying formulas for $x(t), y(t)$.

Part 2. Report a fundamental matrix $\Phi(t)$.

Answers and Solution Details:

Part (a) The details should solve the equation $|A - \lambda I| = 0$ for the three eigenvalues $\lambda = 6, 5, 4$. Then solve the three systems $(A - \lambda I)\vec{v} = \vec{0}$ for eigenvector $\vec{v}$, for $\lambda = 6, 5, 4$.
The eigenpairs are

$$6, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 5, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 4, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Part (b) The answer: $x = c_3 e^{5t} + c_2 t e^{4t} + c_1 e^{4t}$, $y = c_3 e^{5t} + c_2 e^{4t}$, $z = c_3 e^{5t}$.
Solution b(1) Write the system in scalar form

$$\begin{aligned} x' &= 4x + y, \\ y' &= 4y + z, \\ z' &= 5z. \end{aligned}$$

Solve the last equation $z' = 5z$ as
$z = \dfrac{\text{constant}}{\text{integrating factor}} = c_3 e^{5t}$.
$\boxed{z = c_3 e^{5t}}$
The second equation is
$y' = 4y + c_3 e^{5t}$
The linear integrating factor method applies.
$y' - 4y = c_3 e^{-5t}$
$\dfrac{(Wy)'}{W} = c_3 e^{5t}$, where $W = e^{-4t}$,
$(Wy)' = c_3 W e^{5t}$
$(e^{-4t} y)' = c_3 e^{-4t} e^{5t}$
$e^{-4t} y = c_3 e^{t} + c_2$.
$\boxed{y = c_3 e^{5t} + c_2 e^{4t}}$
Stuff the expression into the first differential equation:
$x' = 4x + y = 4x + c_3 e^{5t} + c_2 e^{4t}$
Then solve with the linear integrating factor method.
$x' - 4x = c_3 e^{5t} + c_2 e^{4t}$
$\dfrac{(Wx)'}{W} = c_3 e^{5t} + c_2 e^{4t}$, where $W = e^{-4t}$. Cross-multiply:
$(e^{-4t} x)' = c_3 e^{5t} e^{-4t} + c_2 e^{4t} e^{-4t}$, then integrate:
$e^{-4t} x = c_3 e^{t} + c_2\, t + c_1$
Then divide by e^{-4t}:
$\boxed{x = c_3 e^{5t} + c_2 t e^{4t} + c_1 e^{4t}}$
Solution b(2).
The matrix of coefficients is not diagonalizable, therefore the eigenanalysis method fails to apply.

Part (c) The equations

$$x' = 5x + 4y, \quad y' = -4x + 5y$$

have coefficient matrix $A = \begin{pmatrix} 5 & 4 \\ -4 & 5 \end{pmatrix}$ with characteristic equation $(\lambda-5)^2+16=0$. The roots are $5 \pm 4i$. The Euler atoms are $e^{5t}\cos(4t), e^{5t}\sin(4t)$.

Solution c(1).

By C-H-Z, $x = c_1 e^{5t}\cos(4t) + c_2 e^{5t}\sin(4t)$. Isolate y from the first differential equation $x' = 5x + 4y$, obtaining the formula $4y = x' - 5x = 5x + e^{5t}\left(-4c_1\sin(4t) + 4c_2\cos(4t)\right) - 5x = -4c_1 e^{5t}\sin(4t) + 4c_2 e^{5t}\cos(4t)$. Then the solution formulas are

$$x = c_1 e^{5t}\cos(4t) + c_2 e^{5t}\sin(4t), \quad y(t) = -c_1 e^{5t}\sin(4t) + c_2 e^{5t}\cos(4t).$$

Solution c(2)

A fundamental matrix $\Phi(t)$ is found by taking partial derivatives on the symbols c_1, c_2. The answer is exactly the Jacobian matrix of $\begin{pmatrix} x \\ y \end{pmatrix}$ with respect to variables c_1, c_2.

$\Phi(t) = \begin{pmatrix} e^{5t}\cos(4t) & e^{5t}\sin(4t) \\ -e^{5t}\sin(t) & e^{5t}\cos(t) \end{pmatrix}$.

Use this page to start your solution.

Chapter 6

3. (Linear and Nonlinear Dynamical Systems)

(a) Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node. Sub-classification into improper or proper node is not required.

$$\vec{u}' = \begin{pmatrix} -3 & 1 \\ -2 & 1 \end{pmatrix} \vec{u}$$

(b) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 + 2y + 32, \\ y' &= 2x(x + 2y). \end{aligned}$$

An equilibrium point is $x = -8$, $y = 4$. Compute the Jacobian matrix $A = J(-8, 4)$ of the linearized system at this equilibrium point.

(c) Consider the soft nonlinear spring system $\begin{cases} x' &= y, \\ y' &= -5x - 2y + \frac{5}{4}x^3. \end{cases}$

At equilibrium point $x = 0$, $y = 0$, the Jacobian matrix is $A = J(0, 0) = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$.

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$.

(2) Apply the Pasting Theorem to classify $x = 0$, $y = 0$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

(3) Repeat the classification details of the previous two parts (1), (2) for the other two equilibrium points $(2, 0), (-2, 0)$, for which the Jacobian is $A = J(\pm 2, 0) = \begin{pmatrix} 0 & 1 \\ 10 & -2 \end{pmatrix}$.

Answers and Solution Details:

Part (a) Answer: **unstable saddle**.
It is an unstable saddle. Details: The eigenvalues of A are roots of $r^2+2r-1=0$, which are real roots $a=\sqrt{2}-1, b=-\sqrt{2}-1$ having opposite signs. No rotation eliminates the center and spiral. Finally, the atoms $e^a t$, $e^{bt}t$ have limit infinity, zero at $t=\infty$, therefore the system cannot be a node [nodes have limit $(0,0)$ at one of $t=\infty$ ot $t=-\infty$]. So it must be a saddle.

Part (b) The Jacobian is $J(x,y)=\begin{pmatrix} 0 & 1 \\ -5+\frac{15}{4}x^3 & -2 \end{pmatrix}$. Then $A=J(-8,4)=\begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$.

Part (c)
Solution (1)
The Jacobian is $J(x,y)=\begin{pmatrix} 0 & 1 \\ -5+\frac{15}{4}x^2 & -2 \end{pmatrix}$. Then $A=J(0,0)=\begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$. The eigenvalues of A are found from $r^2+2r+5=0$, giving complex conjugate roots $-1\pm 2i$. Because trig functions appear in the Euler solution atoms, then rotation happens, and the classification must be a center or a spiral. The Euler solution atoms limit to zero at $t=\infty$, therefore it is a spiral and we report a **stable spiral** for the linear problem $\vec{u}'=A\vec{u}$ at equilibrium $\vec{u}=\vec{0}$.

Solution (2)
Theorem 2 in Edwards-Penney section 6.2 applies to say that the same is true for the nonlinear system. Report: **stable spiral** at $x=0$, $y=0$.

Solution (3)
The Jacobian is $J(x,y)=\begin{pmatrix} 0 & 1 \\ -5+\frac{15}{4}x^2 & -2 \end{pmatrix}$. Then $A=J(\pm 2,0)=\begin{pmatrix} 0 & 1 \\ 10 & -2 \end{pmatrix}$. The eigenvalues of A are found from $r^2+2r-10=0$, roots $=-1\pm\sqrt{11}$. The Euler atoms are e^{at}, e^{bt} where a,b have opposite sign. No rotation implies a node or a saddle. Because the atoms limit to $(\infty,0)$ at $t=\infty$, then the node is eliminated and the equilibrium is a saddle. The Pasting Theorem implies the saddle is transferred to the nonlinear phase portrait. Report: unstable saddle.

Use this page to start your solution.