Differential Equations 2280

Sample Midterm Exam 2 with Solutions
Exam Date: 1 April 2016 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exam 1.

1. (Chapter 3)
(a) [50%] Find by any applicable method the steady-state periodic solution for the current equation $I'' + 2I' + 5I = -10\sin(t)$.
(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' = 1 - x$. Show all steps in variation of parameters. Check the answer by quadrature. This problem is moved to Exam 3.

2. (Chapters 1, 2, 3)
(2a) [20%] Solve $2v'(t) = -8 + \dfrac{2}{2t+1}v(t)$, $v(0) = -4$. Show all integrating factor steps.

(2b) [10%] Solve for the general solution: $y'' + 4y' + 6y = 0$.

(2c) [10%] Solve for the general solution of the homogeneous constant-coefficient differential equation whose characteristic equation is $r(r^2 + r)^2(r^2 + 9)^2 = 0$.

(2d) [20%] Find a linear homogeneous constant coefficient differential equation of lowest order which has a particular solution $y = x + \sin\sqrt{2}x + e^{-x}\cos 3x$.

(2e) [15%] A particular solution of the equation $mx'' + cx' + kx = F_0\cos(2t)$ happens to be $x(t) = 11\cos 2t + e^{-t}\sin\sqrt{11}t - \sqrt{11}\sin 2t$. Assume m, c, k all positive. Find the unique periodic steady-state solution x_{SS}.

(2f) [25%] Determine for $y''' + y'' = 100x^2 + \sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

3. (Laplace Theory)
(a) [50%] Solve by Laplace's method $x'' + 2x' + x = e^t$, $x(0) = x'(0) = 0$.
(b) [25%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation

$$\frac{d}{ds}\mathcal{L}(f(t))\Big|_{s\to(s-3)} = \mathcal{L}(f(t) - t).$$

(c) [25%] Derive an integral formula for $y(t)$ by Laplace transform methods, explicitly using the Convolution Theorem, for the problem

$$y''(t) + 4y'(t) + 4y(t) = f(t), \quad y(0) = y'(0) = 0.$$

This is similar to a required homework problem from Chapter 7.

4. (Laplace Theory)
(4a) [20%] Explain Laplace's Method, as applied to the differential equation $x'(t) + 2x(t) = e^t$, $x(0) = 1$. Reference only. Not to appear on any exam.

(4b) [15%] Solve $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+1)(s^2+4)}$ for $f(t)$. [Edited 29 March]

(4c) [15%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s+3)}$.

(4d) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.

(4e) [20%] Solve $x''' + x'' = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$ by Laplace's Method.

(4f) [20%] Solve the system $x' = x + y$, $y' = x - y + 2$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

5. (Laplace Theory)

(a) [30%] Solve $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+s)(s^2-s)}$ for $f(t)$.

(b) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2+4s+5}$.

(c) [20%] Let $u(t)$ denote the unit step. Solve for $f(t)$ in the relation

$$\mathcal{L}(f(t)) = \frac{d}{ds}\mathcal{L}(u(t-1)\sin 2t)$$

Remark: This is not a second shifting theorem problem.

(d) [30%] Compute $\mathcal{L}(e^{2t}f(t))$ for

$$f(t) = \frac{e^t - e^{-t}}{t}.$$

6. (Systems of Differential Equations)

The eigenanalysis method says that, for a 3×3 system $\mathbf{x}' = A\mathbf{x}$, the general solution is $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1,2,3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix},$$

then

(a) [75%] Display eigenanalysis details for A.

(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(c) Repeat (a), (b) for the matrix $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix}$.

7. (Systems of Differential Equations)

(a) [30%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}$.

(b) [20%] Justify that eigenvectors of A corresponding to the eigenvalues in increasing order are the four vectors

$$\begin{pmatrix} 1 \\ -5 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) [50%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Eigenanalysis method. This is identical to the answer

8. (Systems of Differential Equations)

(a) [100%] The eigenvalues are 3, 5 for the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5). Assume initial condition $\vec{u}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.