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Differential Equations 2280

Sample Midterm Exam 1
Exam Date: Friday, 17 February 2017 at 12:50pm

Instructions: This in-class exam is designed for 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

(a) [40%] Solve $y' = \dfrac{2x^3}{1+x^2}$.

(b) [60%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^{-t}v(t)) = 2\,e^t$, $v(0) = 5$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(2) = 2$.

Solution to Problem 1.

(a) Answer $y = x^2 - \ln\left(x^2+1\right) + c$. The integral of $F(x) = \frac{2x^3}{1+x^2}$ is found by substitution $u = 1+x^2$, resulting in the new integration problem $\int F\,dx = \int \frac{u-1}{u} du = \int (1) du - \int \frac{du}{u}$.

(b) Velocity $v(t) = 2e^{2t} + 3e^t$ by quadrature. Integrate $x'(t) = 2e^{2t} + 3e^t$ with $x(0) = 2$ to obtain position $x(t) = e^{2t} + 3e^t - 2$.

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2. (Solve a Separable Equation)
Given $(5y+10)y' = \left(xe^{-x} + \sin(x)\cos(x)\right)(y^2+3y-4)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

Solution to Problem 2.
The solution by separation of variables identifies the separated equation $y' = F(x)G(y)$ using

$$F(x) = xe^{-x} + \sin(x)\cos(x), \quad G(y) = \frac{y^2+3y-4}{5y+10}.$$

The integral of F is done by parts and also by u-substitution.

$$\begin{array}{lcl}
\int F dx &=& \int xe^{-x}\,dx + \int \sin(x)\cos(x)\,dx \\
&=& I_1 + I_2. \\
I_1 &=& \int xe^{-x}\,dx \\
&=& -xe^{-x} - \int e^{-x}\,dx, \quad \text{parts} \quad u = x, dv = e^{-x}dx, \\
&=& -xe^{-x} - e^{-x} + c_1 \\
I_2 &=& \int \sin(x)\cos(x)\,dx \\
&=& \int u\,du, \quad u = \sin(x), du = \cos(x)dx, \\
&=& u^2/2 + c_2 \\
&=& \frac{1}{2}\sin^2(x) + c_2
\end{array}$$

Then $\displaystyle\int F dx = -xe^{-x} - e^{-x} + \frac{1}{2}\sin^2(x) + c_3$.

The integral of $1/G(y)$ requires partial fractions. The details:

$$\begin{array}{lcl}
\displaystyle\int \frac{dx}{G(y(x))} &=& \displaystyle\int \frac{5u+10}{u^2+3u-4}\,du, \quad u = y(x), du = y'(x)dx, \\
&=& \displaystyle\int \frac{5u+10}{(u+4)(u-1)}\,du \\
&=& \displaystyle\int \frac{A}{u+4} + \frac{B}{u-1}\,du, \quad A, B \quad \text{determined later}, \\
&=& A\ln|u+4| + B\ln|u-1| + c_4
\end{array}$$

The partial fraction problem

$$\frac{5u+10}{(u+4)(u-1)} = \frac{A}{u+4} + \frac{B}{u-1}$$

can be solved in a variety of ways, with answer $A = \frac{-20+10}{-5} = 2$ and $B = \frac{15}{5} = 3$. The final implicit solution is obtained from $\int \frac{dx}{G(y(x))} = \int F(x)dx$, which gives the equation

$$2\ln|y+4| + 3\ln|y-1| = -xe^{-x} - e^{-x} + \frac{1}{2}\sin^2(x) + c.$$

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3. (Linear Equations)

(a) [60%] Solve the linear model $2x'(t) = -64 + \dfrac{10}{3t+2}x(t)$, $x(0) = 32$. Show all integrating factor steps.
(b) [20%] Solve $\dfrac{dy}{dx} - (\cos(x))y = 0$ using the homogeneous linear equation shortcut.
(c) [20%] Solve $5\dfrac{dy}{dx} - 7y = 10$ using the superposition principle $y = y_h + y_p$ shortcut. Expected are answers for y_h and y_p.

Solution to Problem 3.
(a) The answer is $v(t) = 32 + 48t$. The details:

$$\begin{aligned}
&v'(t) = -32 + \frac{5}{3t+2}v(t), \\
&v'(t) + \frac{-5}{3t+2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t) \\
&p(t) = \frac{-5}{3t+2}, \\
&W = e^{\int p\,dt}, \quad \text{integrating factor} \\
&W = e^u, \quad u = \textstyle\int p\,dt = -\frac{5}{3}\ln|3t+2| = \ln\left(|3t+2|^{-5/3}\right) \\
&W = (3t+2)^{-5/3}, \quad \text{Final choice for } W.
\end{aligned}$$

Then replace the left side of $v' + pv = q$ by $(vW)'/W$ to obtain

$$\begin{aligned}
&v'(t) + \frac{-5}{3t+2}v(t) = -32, \quad \text{standard form } v' + p(t)v = q(t) \\
&\frac{(vW)'}{W} = -32, \quad \text{Replace left side by quotient } (vW)'/W \\
&(vW)' = -32W, \quad \text{cross-multiply} \\
&vW = -32\textstyle\int W\,dt, \quad \text{quadrature step.}
\end{aligned}$$

The evaluation of the integral is from the power rule:

$$\int -32W\,dt = -32\int (3t+2)^{-5/3}dt = -32\frac{(3t+2)^{-2/3}}{(-2/3)(3)} + c.$$

Division by $W = (3t+2)^{-5/3}$ then gives the general solution

$$v(t) = \frac{c}{W} - \frac{32}{-2}(3t+2)^{-2/3}(3t+2)^{5/3}.$$

Constant c evaluates to $c = 0$ because of initial condition $v(0) = 32$. Then

$$v(t) = \frac{32}{-2}(3t+2)^{-2/3}(3t+2)^{5/3} = 16(3t+2)^{-\frac{2}{3}+\frac{5}{3}} = 16(3t+2).$$

(b) The answer is $y =$ constant divided by the integrating factor: $y = \dfrac{c}{W}$. Because $W = e^u$ where $u = \int -\cos(x)dx = -\sin x$, then $y = ce^{\sin x}$.
(c) The equilibrium solution (a constant solution) is $y_p = -\frac{10}{7}$. The homogeneous solution is $y_h = ce^{7x/5} =$ constant divided by the integrating factor. Then $y = y_p + y_h = -\frac{10}{7} + ce^{7x/5}$.

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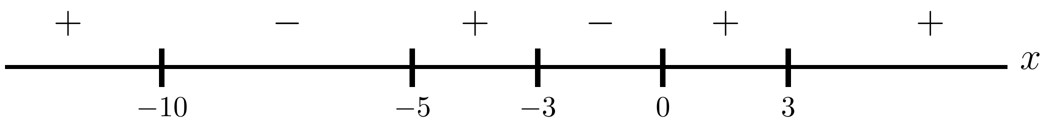
4. (Stability)

Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase portrait with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.



Solution to Problem 4.
The graphic is drawn using increasing and decreasing curves, which may or may not be depicted with turning points. The rules:

1. A curve drawn between equilibria is increasing if the sign is PLUS.
2. A curve drawn between equilibria is decreasing if the sign is MINUS.
3. Label: FUNNEL, STABLE
   The signs left to right are PLUS MINUS crossing the equilibrium point.
4. Label: SPOUT, UNSTABLE
   The signs left to right are MINUS PLUS crossing the equilibrium point.
5. Label: NODE, UNSTABLE
   The signs left to right are PLUS PLUS crossing the equilibrium point, or
   The signs left to right are MINUS MINUS crossing the equilibrium point.

The answer:

$x = -10$: FUNNEL, STABLE
$x = -5$: SPOUT, UNSTABLE
$x = -3$: FUNNEL, STABLE
$x = 0$: SPOUT, UNSTABLE
$x = 3$: NODE, UNSTABLE

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5. (ch3)

Using Euler's theorem on Euler solution atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c).

(a) [40%] Find a constant coefficient differential equation $ay'' + by' + cy = 0$ which has particular solutions $-5e^{-x} + xe^{-x}$, $10e^{-x} + xe^{-x}$.

(b) [30%] Given characteristic equation $r(r-2)(r^3+4r)(r^2+2r+37) = 0$, solve the differential equation.

(c) [30%] Given $mx''(t) + cx'(t) + kx(t) = 0$, which represents an unforced damped spring-mass system. Assume $m = 4$, $c = 4$, $k = 129$. Classify the equation as over-damped, critically damped or under-damped. Illustrate in a spring-mass-dashpot drawing the assignment of physical constants m, c, k and the initial conditions $x(0) = 1$, $x'(0) = 0$.

Solution to Problem 5.

5(a)

Multiply the first solution by 2 and add it to the second solution. Then Euler atom xe^{-x} is a solution, which implies that $r = -1$ is a double root of the characteristic equation. Then the characteristic equation should be $(r-(-1))(r-(-1)) = 0$, or $r^2 + 2r + 1 = 0$. The differential equation is $y'' + 2y' + y = 0$.

5(b)

The characteristic equation factors into $r^2(r-2)(r^2+4)((r+1)^2+36) = 0$ with roots $r = 0, 0; 2; \pm 2i; -1 \pm 6i$. Then y is a linear combination of the Euler solution atoms:

$1, x, e^{2x}, \cos(2x), \sin(2x); e^{-x}\cos(6x), e^{-x}\sin(6x)$.

5(c)

Use $4r^2 + 4r + 129 = 0$ and the quadratic formula to obtain roots $r = -1/2 + 4\sqrt{2}i, -1/2 - 4\sqrt{2}i$ and Euler solution atoms $e^{-x/2}\cos 4\sqrt{2}t, e^{-x/2}\sin 4\sqrt{2}t$. Then y is a linear combination of these two solution atoms, and it oscillates, therefore the classification is **under-damped**. The illustration shows a spring, a dashpot and a mass with labels k, c, m. Initial conditions mean mass elongation $x = 1$, at rest.

A **dashpot** is represented as a cylinder and piston with rod, the rod attached to the mass. Variable x is positive in the down direction and negative in the up direction. The equilibrium position is $x = 0$.