11.7 Nonhomogeneous Linear Systems

Variation of Parameters

The method of variation of parameters is a general method for solving a linear nonhomogeneous system

\[ x' = Ax + F(t). \]

Historically, it was a trial solution method, whereby the nonhomogeneous system is solved using a trial solution of the form

\[ x(t) = e^{At}x_0(t). \]

In this formula, \( x_0(t) \) is a vector function to be determined. The method is imagined to originate by varying \( x_0 \) in the general solution \( x(t) = e^{At}x_0 \) of the linear homogenous system \( x' = Ax \). Hence was coined the names variation of parameters and variation of constants.

Modern use of variation of parameters is through a formula, memorized for routine use.

**Theorem 28 (Variation of Parameters for Systems)**

Let \( A \) be a constant \( n \times n \) matrix and \( F(t) \) a continuous function near \( t = t_0 \). The unique solution \( x(t) \) of the matrix initial value problem

\[ x'(t) = Ax(t) + F(t), \quad x(t_0) = x_0, \]

is given by the variation of parameters formula

\[ x(t) = e^{At}x_0 + e^{At} \int_{t_0}^{t} e^{-rA}F(r)dr. \]

**Proof of (1).** Define

\[ u(t) = x_0 + \int_{t_0}^{t} e^{-rA}F(r)dr. \]

To show (1) holds, we must verify \( x(t) = e^{At}u(t) \). First, the function \( u(t) \) is differentiable with continuous derivative \( e^{-tA}F(t) \), by the fundamental theorem of calculus applied to each of its components. The product rule of calculus applies to give

\[ x'(t) = (e^{At})' u(t) + e^{At}u'(t) = Ae^{At}u(t) + e^{At}e^{-tA}F(t) = A(x(t) + F(t)). \]

Therefore, \( x(t) \) satisfies the differential equation \( x' = Ax + F(t) \). Because \( u(t_0) = x_0 \), then \( x(t_0) = x_0 \), which shows the initial condition is also satisfied. The proof is complete.
Undetermined Coefficients

The trial solution method known as the method of undetermined coefficients can be applied to vector-matrix systems \( \mathbf{x}' = A \mathbf{x} + \mathbf{F}(t) \) when the components of \( \mathbf{F} \) are sums of terms of the form

\[
(polynomial \ in \ t)e^{at}(\cos(bt) \ or \ \sin(bt)).
\]

Such terms are known as **atoms**. It is usually efficient to write \( \mathbf{F} \) in terms of the columns \( \mathbf{e}_1, \ldots, \mathbf{e}_n \) of the \( n \times n \) identity matrix \( \mathbf{I} \), as the combination

\[
\mathbf{F}(t) = \sum_{j=1}^{n} F_j(t) \mathbf{e}_j.
\]

Then

\[
\mathbf{x}(t) = \sum_{j=1}^{n} \mathbf{x}_j(t),
\]

where \( \mathbf{x}_j(t) \) is a particular solution of the simpler equation

\[
\mathbf{x}'(t) = A\mathbf{x}(t) + f(t)\mathbf{c}, \quad f = F_j, \quad \mathbf{c} = \mathbf{e}_j.
\]

An initial trial solution \( \mathbf{x}(t) \) for \( \mathbf{x}'(t) = A\mathbf{x}(t) + f(t)\mathbf{c} \) can be determined from the following **initial trial solution rule**:

Assume \( f(t) \) is a sum of atoms. Identify independent functions whose linear combinations give all derivatives of \( f(t) \). Let the initial trial solution be a linear combination of these functions with undetermined vector coefficients \( \{c_j\} \).

In the well-known scalar case, the trial solution must be modified if its terms contain any portion of the general solution to the homogeneous equation. If \( f(t) \) is a polynomial, then the **correction rule** for the initial trial solution is avoided by assuming the matrix \( A \) is invertible. This assumption means that \( r = 0 \) is not a root of \( \det(A - r\mathbf{I}) = 0 \), which prevents the homogenous solution from having any polynomial terms.

The initial vector trial solution is substituted into the differential equation to find the undetermined coefficients \( \{c_j\} \), hence finding a particular solution.

**Theorem 29 (Polynomial solutions)**

Let \( f(t) = \sum_{j=0}^{k} p_j t^j \) be a polynomial of degree \( k \). Assume \( A \) is an \( n \times n \) constant invertible matrix. Then \( \mathbf{u}' = A\mathbf{u} + f(t)\mathbf{c} \) has a polynomial solution \( \mathbf{u}(t) = \sum_{j=0}^{k} c_j t^j \) of degree \( k \) with vector coefficients \( \{c_j\} \) given by the relations

\[
c_j = -\sum_{i=j}^{k} p_i A^{i-j-1} \mathbf{c}, \quad 0 \leq j \leq k.
\]
Theorem 30 (Polynomial × exponential solutions)
Let \( g(t) = \sum_{j=0}^{k} p_j \frac{t^j}{j!} \) be a polynomial of degree \( k \). Assume \( A \) is an \( n \times n \) constant matrix and \( B = A - aI \) is invertible. Then \( u' = Au + e^{at}g(t)c \) has a polynomial-exponential solution \( u(t) = e^{at} \sum_{j=0}^{k} c_j \frac{t^j}{j!} \) with vector coefficients \( \{c_j\} \) given by the relations

\[
c_j = -\sum_{i=j}^{k} p_i B^{i-j} c_{i-j}, \quad 0 \leq j \leq k.
\]

Proof of Theorem 29. Substitute \( u(t) = \sum_{j=0}^{k} c_j \frac{t^j}{j!} \) into the differential equation, then

\[
\sum_{j=0}^{k-1} c_{j+1} \frac{t^j}{j!} = A \sum_{j=0}^{k} c_j \frac{t^j}{j!} + \sum_{j=0}^{k} p_j \frac{t^j}{j!} c.
\]

Then terms on the right for \( j = k \) must add to zero and the others match the left side coefficients of \( t^j/j! \), giving the relations

\[
A c_k + p_k c = 0, \quad c_{j+1} = A c_j + p_j c.
\]

Solving these relations recursively gives the formulas

\[
\begin{align*}
c_k &= -p_k A^{-1} c, \\
c_{k-1} &= -(p_{k-1} A^{-1} + p_k A^{-2}) c, \\
& \vdots \\
c_0 &= -(p_0 A^{-1} + \cdots + p_k A^{-k-1}) c.
\end{align*}
\]

The relations above can be summarized by the formula

\[
c_j = -\sum_{i=j}^{k} p_i B^{i-j} c_{i-j}, \quad 0 \leq j \leq k.
\]

The calculation shows that if \( u(t) = \sum_{j=0}^{k} c_j \frac{t^j}{j!} \) and \( c_j \) is given by the last formula, then \( u(t) \) substituted into the differential equation gives matching LHS and RHS. The proof is complete.

Proof of Theorem 30. Let \( u(t) = e^{at}v(t) \). Then \( u' = Au + e^{at}g(t)c \) implies \( v' = (A - aI)v + g(t)c \). Apply Theorem 29 to \( v' = Bv + g(t)c \). The proof is complete.