Applications of Linear Algebra in Political Science: Candidate Preferences in Elections

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In democratic states with liberal elections, voters almost always have multiple candidates to choose from. These candidates typically represent different political parties, ideologies, or voting pacts. Voters have transitive preferences for these multiple candidates. Due to a variety of factors, an individual's most preferred candidate may not be elected. Electoral rules are often responsible for diverging voting preferences from actual votes cast. Individuals often know that their vote actually counting is contingent on how others vote and electoral rules. As a result, individuals anticipate such electoral rules and potential for certain candidates and adjust their actual vote (which may depart from the individual's true preference). Linear algebra has been applied to the problem of voting preferences in the past to determine ways that a voter (or group of voters) can change their behavior to elect certain preferred candidates.

Linear Algebra in Election Preferences

This project will investigate and report on the mathematics used to analyze the preferences of voters for candidates in elections. The voting layout is put into a majority cycle, however, since these cycles are complicated to read and understand. They can be deconstructed into subsets that represent an individuals' preferences. These cycles are then added to linear combinations, which is where linear algebra comes into the equation. The majority cycle is decomposed into a cyclic and acyclic vectors. Through manipulation of the vectors, we can find the basis of the linear combination. This project will discuss, compute and analyze each of these aspects of linear algebra, and how they are applied to the voting system. The reason mathematicians found interest in creating these voting paradoxes is due to the easy manipulation of results based on the desired outcome of the person conducting the vote.

Example Linear Algebra in Election Preferences

In American elections, people have two primary choices and usually a third independent choice. People tend to have intransitive ordered preferences.

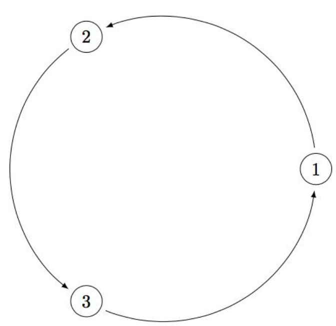
Preference order	*Number with that preference*
Democrat > Republican > Third	3
Democrat > Third > Republican	3
Republican > Democrat > Third	5
Republican > Third > Democrat	7

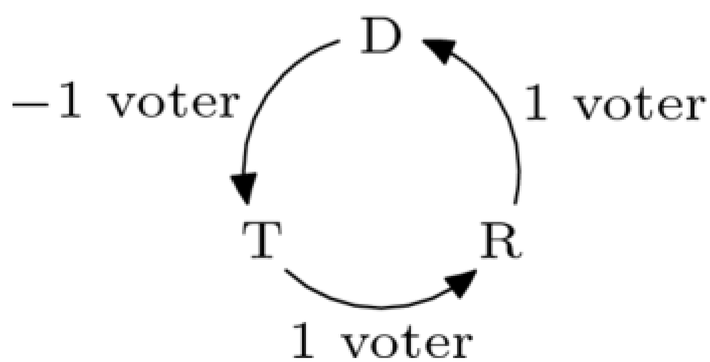
Third > Democrat > Republican	2
Third > Republican > Democrat	1

Total: 21

The image below represents the cyclical voting preferences of each individual. The total system then can be described by adding the total number of individuals who prefer candidate 1 (Democrat) to 2 (Republican), 2 to 3 (third), and 3 to 1.

-5 people prefer Democrat to Republican (1 to 2)
-9 people prefer Republican to Third (2 to 3)
-1 people prefer Third to Democrat (3 to 1)





These preferences can then be broken down into individual preferences (right image, sourced from http://joshua.smcvt.edu/linearalgebra/book.pdf). The intuition behind this is that this voter actually prefers the third candidate to the Democratic candidate (hence the negative sign). This model can be used to represent all voters individual preferences. To get the total model, there is both a cyclical and acyclical model. The cyclical preferences are then added together (i.e., a linear combination). Our overall system can be represented by the following equation:

(1)
$D_1 * \langle -1, 1, 1 \rangle + D_2 * \langle -1, -1, 1 \rangle + D_3 * \langle 1, -1, -1 \rangle\ b$, where $b = \langle -5, -9, -1 \rangle$

But is there another way to represent voter preferences? Next, we will try to decompose a single vote vector <-1,1,1> into two parts: cyclic and acyclic systems. A vector in R^3 is purely cyclic if Vector C $= \langle a, a, a \rangle$ given that a is part of R (all real numbers).

To find the acyclic system, we need to find C perp, where:
(2)
C perp = $\langle C_1, C_2, C_3 \rangle * \langle a, a, a \rangle = 0$

The basis can be found by first finding:

(3)

$C_2 * \ <-1,1,0> \ + C_3 * \ <-1,0,1>$ and $a *<1,1,1> \ = C$

Thus, the basis is:

(4)

$<-1,1,0>, \ <-1,0,1>, \ and <1,1,1>.$

The vector solution with Strang's special solution to an individual voter are:

(5)

$C_1 - C_2 - C_3 = -1$

$C_1 + C_2 = 1$

$C_1 + C_3 = 1$

(6)

$C_1 = 1/3$

$C_2 = 2/3$

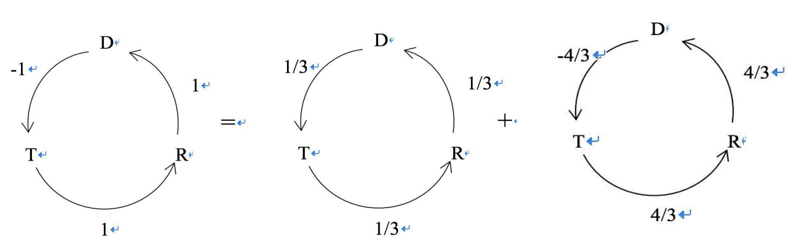
$C_3 = 2/3$

This suggests that

(7)

$<-1,1,1> \ = (1/3) \ *<1,1,1> \ + (2/3) *<-1,1,0> \ + (2/3) *<-1,0,1> \ = \ <1/3,1/3,/1,3> \ + \ <-4/3,4/3,4/3>$

Giving us both the cyclical and acyclical components. This can be replicated for the 3 types of voter preferences, and by flipping signs, would also allow us to find the negative spins of each voter preference. Finding the overall preference is then just a matter of adding up these cyclical preferences.



Conclusion

As mentioned above, these voting paradoxes have pointed to a fault in the voting system due to the easy manipulation of the vote itself. Depending on the order the party is asked in, the desired result easily follows. This example shows how this voting process, along with nearly every other voting process is not entirely fair, due to a lack of protection from biases.