Image Compression with Haar Wavelets

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Image Compression Process

- Separate an image into 8×8 sub-matrices, A_i.
- Build an orthonormal basis, U, from the Haar matrix.
- Perform the Haar transform on each sub-matrix, $B_i = U^{-1} A_i U$.
- Set elements of B_i to 0 with a threshold value, ϵ.
- Reverse the transform and rebuild the image.

General Haar Matrix

The $2n \times 2n$ Haar matrix is defined to be

$$H_{2n} = \begin{bmatrix} H_n \otimes [1, 1] \\ I_n \otimes [1, -1] \end{bmatrix},$$

where $\otimes$ is the Kronecker product and I_n is an $n \times n$ identity matrix.

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},\ H_4 = \begin{bmatrix} 1[1,1] & 1[1,1] \\ 1[1,1] & -1[1,1] \\ 1[1,-1] & 0[1,-1] \\ 0[1,-1] & 1[1,-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

8×8 Haar Matrix

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Let $H_8^{\mathsf{T}} = [\, \mathbf{h}_1 \ \mathbf{h}_2 \ \dots \ \mathbf{h}_8 \,]$. Notice that $\mathbf{h}_i \cdot \mathbf{h}_j = 0$ for all $i \neq j$. In other words, the columns of H_8^{T} are pairwise orthogonal.

Orthonormal Basis

We can build an orthonormal basis, U, by simply normalizing the columns of H_8^T. Hence,

$$U = \begin{bmatrix} \frac{\mathbf{h}_1}{\|\mathbf{h}_1\|} & \frac{\mathbf{h}_2}{\|\mathbf{h}_2\|} & \dots & \frac{\mathbf{h}_8}{\|\mathbf{h}_8\|} \end{bmatrix}$$

$$= \begin{bmatrix}
\frac{\sqrt{8}}{8} & \frac{\sqrt{8}}{8} & \frac{1}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\
\frac{\sqrt{8}}{8} & \frac{\sqrt{8}}{8} & \frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\
\frac{\sqrt{8}}{8} & \frac{\sqrt{8}}{8} & -\frac{1}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\
\frac{\sqrt{8}}{8} & \frac{\sqrt{8}}{8} & -\frac{1}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\
\frac{\sqrt{8}}{8} & -\frac{\sqrt{8}}{8} & 0 & \frac{1}{2} & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{8}}{8} & -\frac{\sqrt{8}}{8} & 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{8}}{8} & -\frac{\sqrt{8}}{8} & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{8}}{8} & -\frac{\sqrt{8}}{8} & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{\sqrt{2}}{2}
\end{bmatrix}.$$

Since the columns of U are orthonormal and U is square, it follows that $UU^\mathsf{T} = U^\mathsf{T}U = I$, i.e. $U^{-1} = U^\mathsf{T}$.

Haar Transform

With a sub-matrix, A, and the orthonormal basis, U, we can now perform the Haar transform. The transform is defined as, $B = U^{-1}AU$ with elements $b_{i,j}$.

- Set elements, $b_{i,j}$, of B to 0 to compress the image.
- If the absolute value of an element, $b_{i,j}$, is less than a chosen ϵ, the element is set to 0.
- Let $\hat{B}$ be the matrix that follows this condition,

$$\hat{B} = \left\{ b_{i,j} \in B \;\middle|\; |b_{i,j}| < \epsilon \implies b_{i,j} = 0 \right\}.$$

Reverse the Haar Transform

To finish the image compression we need to reverse the Haar transform on $\hat{B}$.

$$\begin{aligned}
\text{Recall, } B &= U^{-1}AU \\
\text{Then, } UBU^{-1} &= UU^{-1}AUU^{-1} \\
&= IAI \\
&= A \\
\text{Thus, } \hat{A} &= U\hat{B}U^{-1}
\end{aligned}$$

Rounding the elements of $\hat{A}$ to unsigned integers completes the process.

Examples

$\epsilon = 0$ (64.9 kB)

$\epsilon = 25$ (42.2 kB)

Examples (continued)

$\epsilon = 50$ (29.9 kB)

$\epsilon = 100$ (21.2kB)