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Semester Project

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The Spectral Differences in Sounds and Harmonics

Through chemical and biological processes, we, as humans, can detect the movement and vibrations of molecules through the air and interpret them as sound. That dog barking in the distance, the light whistle of the wind through the window, the rich sound of a cello playing a long, deep note or a friend laughing in the other room. Sound is all around us. This continuous array of noise is often seen as disordered, muddled, and random. However, just as snowflakes are constructed with delicate patterns of ice crystals, sound is very formulaic and structured in its makeup. And as such, through linear algebra, we are able to measure it and compare it.

A sound spectrum is a representation of the amount of molecular vibration that occurs at each frequency over time. Each sound has its own unique spectrum and frequency. For example, the sound of Middle C on a standard grand piano vibrates at 261.6 Hertz (cycles per second). A0 is 27.50 Hertz. The high pitch ringing from an old TV is about 15 kiloHertz, lying just within the average adult human's range of hearing (20 Hz – 18 kHz). The more cycles per second, or rather the faster the molecular vibration that occurs as a result of the hammer hitting the piano string, the higher the pitch of the sound is. Thinner strings are able to vibrate faster than thicker ones, thus producing more vibration back and forth. The intensity, or loudness of a sound is measured in decibels (dB) and is relative to the amplitude of the sound wave.

Our ears use a complex process to detect these signals. After the sound waves enter our ear canal, they strike the tympanic membrane or ear drum within the middle ear, causing it to vibrate. These vibrations are transferred to the cochlea, a 3 cm long tube filled with liquid in the inner ear. The cochlea contains the basilar membrane, which supports over 12,000 sensory cells which make up the cochlear nerve. This nerve is designed to act as a sound spectrum analyzer. When it detects sounds of a higher frequency, the cells near the front of the basilar membrane resonate, and as the frequency gets lower and lower, the corresponding cells that get triggered are deeper and deeper within the ear. The basilar membrane then encodes these vibrations into electrical impulses called action potentials. “For example, a 200 hertz sound wave can be represented by a neuron producing 200 action potentials per second.” - Steven W. Smith, Ph.D. This is known as the volley principle. The electrical impulses are sent to the brain and converted to the sound we hear. Our ears are built to react to the changes in amplitude and spectral densities. But measuring the spectral differences and complexities heard from a human ear is complicated. Mathematically, it is more based on the spectral representation of the two signals and measuring their differences. To better realize what this means, we must employ the use of Fourier analysis.

Fourier Analysis is the study of the way complicated functions, namely Fourier series, can be represented by sums of trigonometric functions. Named after Joseph Fourier, Fourier series are wave-like functions, such as sound spectra, that can be estimated as the sum of multiple sine waves. The function is called a trigonometric polynomial and it is connected to other functions within the basis so long that for any $n \geq 1$, the set of polynomial expressions in the function is

$$\langle f(t), g(t) \rangle = \int_0^L f(t)g(t) dt$$

Figure 1: The Inner Product Formula.

orthogonal with respect to the basis' inner product, shown in **Figure 1** below. This inner product multiplied by $\frac{1}{\pi}$ gives us a function that is orthonormal on the interval $[-\pi, \pi]$. We use this to solve the vector space of functions that make up the Fourier decomposition. Finally, we are able to express this continuous functions as a Fourier Series: $f(x) = a + \sum_{n=1}^{\infty} b_n \cos nx + \sum_{n=1}^{\infty} c_n \sin nx$ Fourier Analysis plays a big role in understanding the spectral density, and the level of energy emitted by the sound.

To better illustrate the concept of Fourier Analysis, I have devised a small experiment. I have a friend that is a music producer, and he swears by the fact that natural instruments sound better, clearer, and purer than a computer generated sound. As I established before, human ears are very complex and are able to pick up and distinguish very slight differences in sound, so I do believe he may have a valid point. I figured that there had to be some sort of difference in the spectral density and amplitude of a computer generated sound and a real sound. But I want to take that hypothesis to the next level and analyze the spectral patterns of a real instrument's note (in this experiment, we will use Middle C) and compare it with the same note of the same type of instrument, only this time it's generated by a computer application. I want to see whether the makeup and pattern of one spectrum has a unique mathematical pattern than the other and if this difference can be quantified.

As you can see in **Figure 2**, the spectrum of both notes are very different from each other. The real trumpet has much more variation and a disturbance in energy across the spectrum, while the computer generated sound is much more uniform and

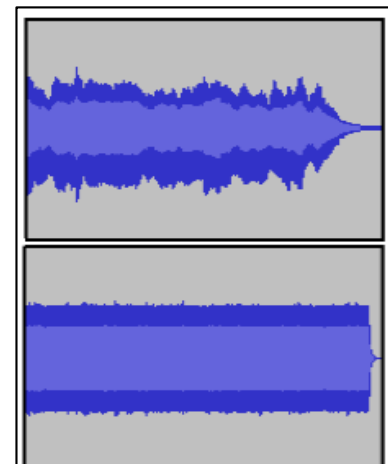
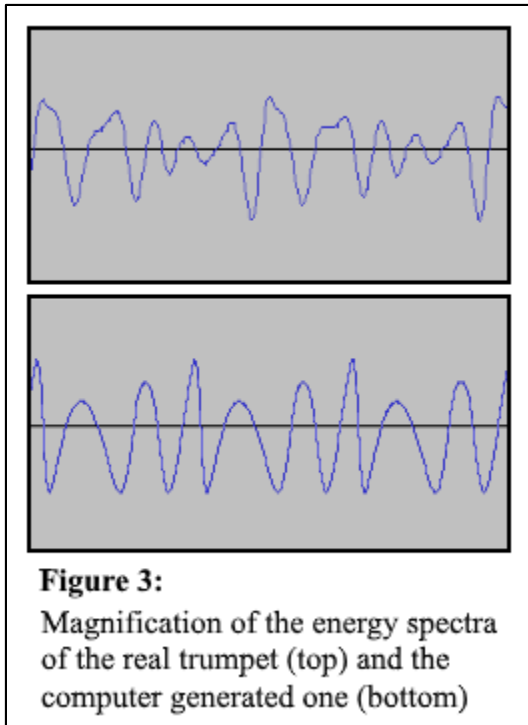


Figure 2:
Notice the difference in uniformity between the real trumpet (top) and the computer generated one (bottom)

monotone. One could argue that the fluctuations in energy is due to the slight palpitations or imperfections that comes from a human blowing into an instrument. And while that may be able to partially explain the wider changes in amplitude, it would do us well to examine the waves at a much closer magnification. See **Figure 3** below. The waves appear to be strikingly similar. The



major difference being the tiny fluctuations in the real trumpet's sound. The computer generated trumpet follows a much more algorithmic and mathematical pattern, as is expected. But both waves are generally similar. The difference is subtle, but through this experiment we have proved that the energy spectra are indeed different.

The point we can conclude from this is that computer generated sounds seem to be pure harmonic. The frequency, amplitude, and spectral

density are all calculated and limited to a simple harmonic formula. And while natural sounds are similar, they appear to be superpositions of several harmonics. Using the Fourier series we derived earlier, we are able to calculate the inner product spaces of all the harmonics that are orthonormal within the interval $[-\pi, \pi]$, which can help quantify the harmonics with respect to a more familiar spectrum. But Linear Algebra can offer us an even clearer meaning. The answer lies in the application of a difference equation. As we have established before, sound is just a signal. And signals can be better analyzed and compared within their orthonormal family. Similar to the Fourier series, mentioned before, in order to find out if a signal has an orthonormal family, we must determine if it is linearly independent. In order to do that, the signal must satisfy

the equation $c_1u_k + c_2v_k + c_3w_k = 0$ for all k . In other words, $c_1 = c_2 = c_3 = 0$. For a set of three signals $\{u_k\}$, $\{v_k\}$, and $\{w_k\}$ we can generate the coefficient matrix:

$$\begin{bmatrix} u_k & v_k & w_k \\ u_{k+1} & v_{k+1} & w_{k+1} \\ u_{k+2} & v_{k+2} & w_{k+2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for all } k$$

This is also known as a Casorati matrix. It was developed by Felice Casorati and is a very useful tool in studying and analyzing linear difference equations. This, along with generating a Fourier Series from the spectral information of a sound helps us visualize and understand the differences in two harmonics. Whether it be a comparison between a computer generated sound and a real one, such as the experiment I conducted, or the fundamental variations of two entirely different sounds, they can all be measured and quantified.

After studying and analyzing energy spectra and differences between them by working through the linear systems provided by Linear Algebra, one gains a much deeper understanding of the concept and mathematics that are at work within a simple sound. They may seem chaotic and unordered, but every sound and signal is a formula that can be analyzed and compared. And through the experiments I've performed, I have demonstrated the differences and similarities within two seemingly identical sounds. With the help of Linear Algebra and modern computational tools, there will only be more progress in this fascinating area.

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