Math 2270-2 Spring 2012
Computer Lab 4
Fourier Series

This lab was written by Professor Chris Cashen in his Postdoc years at Utah. Some minor display, maple code display, and font size changes were made. It remains exactly the same as Chris Cashen wrote.
Submit your project on the due date in class as a worksheet print. **Submit study group** efforts as one worksheet print with multiple names.

See Section 8.5 of the text for problem background.

1 Introduction

Let $\bar{v} =< v_1, v_2, v_3 >$ be a vector in $\mathcal{R}^3$. The vectors $\vec{\imath} =< 1, 0, 0 >$, $\vec{\jmath} =< 0, 1, 0 >$ and $\vec{k} =< 0, 0, 1 >$ are an orthonormal basis for $\mathcal{R}^3$, and $\vec{v}$ can be expressed as a linear combination of these vectors as

$$\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$$

Similarly, for any orthonormal basis $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ we can write $\vec{v}$ as a linear combination

$$\vec{v} = (\vec{v} \cdot \vec{a}_1)\vec{a}_1 + (\vec{v} \cdot \vec{a}_2)\vec{a}_2 + (\vec{v} \cdot \vec{a}_3)\vec{a}_3$$

More generally still, we can do this for any orthogonal basis using projection formulae. If $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ form a basis of orthogonal vectors, we can write $\vec{v}$ as a linear combination

$$\vec{v} = \left(\frac{\vec{v} \cdot \vec{a}_1}{\vec{a}_1 \cdot \vec{a}_1}\right)\vec{a}_1 + \left(\frac{\vec{v} \cdot \vec{a}_2}{\vec{a}_2 \cdot \vec{a}_2}\right)\vec{a}_2 + \left(\frac{\vec{v} \cdot \vec{a}_3}{\vec{a}_3 \cdot \vec{a}_3}\right)\vec{a}_3$$

In this lab we will be interested in rewriting vectors in terms of an orthogonal basis, but the vector space will consist of 2π–periodic functions. You can verify that this is a vector space: it is closed under linear combinations and the zero function serves as the zero vector. The first order of business will be to find a basis. It turns out that this vector space is infinite dimensional.

2 An Orthogonal Basis

We first define an inner product of vectors to take the place of the usual dot product of $\mathcal{R}^n$. If f and g are two functions in the vector space, define their inner product to be:

$$(f, g) = \int_0^{2\pi} f(x)g(x)dx$$

Now we claim that a basis for the vector space is given by the following infinite list of functions:

$$\begin{array}{ll} \cos(mx) & \text{for } m = 0, 1, 2, \ldots \\ \sin(nx) & \text{for } n = 1, 2, \ldots \end{array}$$

First, these "vectors" are all 2π–periodic functions, so they are all in the vector space. They are all non-zero functions (which is why we have not included $\sin(0x)$).
They are linearly independent - in fact, they form an orthogonal set. To verify this we check that if you take the inner product of any two of them you get 0. You may remember these formulae from a table of integrals from Calculus II.

$$\begin{array}{ll} \int_0^{2\pi} \cos(mx)\cos(nx)dx = 0 & \text{for } m \neq n \\ \int_0^{2\pi} \sin(mx)\sin(nx)dx = 0 & \text{for } m \neq n \\ \int_0^{2\pi} \cos(mx)\sin(nx)dx = 0 & \end{array}$$

They are, however, not unit length.

$$\begin{array}{ll} \int_0^{2\pi} \cos(0x)\cos(0x)dx = \int_0^{2\pi} 1 \cdot 1dx = 2\pi & \\ \int_0^{2\pi} \cos(mx)\cos(mx)dx = \pi & \text{for } m \neq 0 \\ \int_0^{2\pi} \sin(nx)\sin(nx)dx = \pi & \text{for } n \neq 0 \end{array}$$

Verify these six formulae in Maple. To get the integrals to evaluate we must make assumptions on the parameters m and n.

```
assume(m>0,m::integer,n>0,n::integer,m!=n);
int(cos(m*x)*sin(n*x), x=0..2Pi);
```

The fact that they are a spanning set for a vector space of 2π–periodic functions is Fourier's Theorem. There are some technicalities here that we will ignore. The complications are due to the fact that the vector space is infinite dimensional, so we will express a function as an infinite series of functions, and we must worry about conditions under which such a series converges. It is true that for *well-behaved* 2π–periodic functions the series will converge.

3 Expressing a Vector in Terms of the Basis

Given a *nice* 2π–periodic function $f(x)$, we want to express it in terms of the basis of sine and cosine functions. For each $m = 0, 1, \ldots$ and $n = 1, 2, \ldots$ we need to find the $\cos(mx)$ and $\sin(nx)$ parts of f. We do this with the usual projection formula.
For any $m = 0, 1, \ldots,$ the $\cos(mx)$ part of f is $a_m \cos(mx)$ where

$$a_m = \frac{(f, \cos(mx))}{(\cos(mx), \cos(mx))} = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} f \cos(mx)dx & \text{if } m = 0 \\ \frac{1}{\pi} \int_0^{2\pi} f \cos(mx)dx & \text{if } m > 0 \end{cases}$$

Similarly, for $n = 1, 2, \ldots,$ the $\sin(nx)$ part of f is $b_n \sin(nx)$ where

$$b_n = \frac{(f, \sin(nx))}{(\sin(nx), \sin(nx))} = \frac{1}{\pi} \int_0^{2\pi} f \sin(nx)dx$$

Then the function f can be written as a sum of its parts as

$$f(x) \approx a_0 \cos(0x) + a_1 \cos(1x) + b_1 \sin(1x) + a_2 \cos(2x) + b_2 \sin(2x) + \ldots$$

Here is an example. Let $f(x)$ be the 2π–periodic function whose value is π between 0 and π and $-\pi$ from π to 2π. In Maple, define this piecewise so that it is correct on the interval from -2π to 2π and plot it:

```
f:= x-> piecewise(x<=-Pi ,Pi,x<0,-Pi,x<=Pi,Pi,-Pi);
plot(f(x), x=-2*Pi..2*Pi, discont=false, thickness=3);
```

The graph of f is described as a **square wave**.
The sine parts we find by computing

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx$$

The fact that f is an odd function implies that all the cosine parts are 0. Verify this.
So, we claim that

$$f(x) \approx \frac{4}{1}\sin(1x) + \frac{4}{3}\sin(3x) + \frac{4}{5}\sin(5x) + \ldots$$

You may remember from Calculus II that the Taylor Series expresses a function as a series of powers of x, and that the series can be approximated by taking a Taylor Polynomial consisting of finitely many terms. We have the same idea here: we have a Fourier Series converging to a function representing f, and we can stop after finitely many terms to get a Fourier Sum approximating f. The more terms we include the better the approximation will be. Try plotting the first four Fourier Sums on the same axes with f:

```
plot([ f(x),4*sin(x)], x=-2*Pi..2*Pi, discont=false,
 color=[red,blue], thickness=[3,1]);
plot([ f(x),4*(sin(x)+(1/3)*sin(3*x))],x=-2*Pi..2*Pi,
 discont=false, color=[red,blue], thickness=[3,1]);
```

Let's parse this Maple command: the first argument of the `plot` function is a list, in square brackets, of 2 functions to plot, separated by commas. The next argument says to display the plot on x-values from -2π to 2π (the y–values will be determined automatically). The remaining arguments specify options. `discont=false` says to draw the graphs as if they were continuous, even though f, for instance, is not continuous. `color` and `thickness` define these attributes for the graphs of the corresponding functions. The first function from the list will be graphed with color red and thickness 3, etc.

4 Energy Spectrum

The k–th harmonic of f is the frequency $\frac{k}{2\pi}$ part of the Fourier Series: $a_k\cos(kx) + b_k\sin(kx)$.
The *energy of the k-th harmonic* is $a_k^2+b_k^2$. There is an Energy Theorem that says that the energy of a function is equal to the sum of the energies of the harmonics. Since the Fourier Series converges, these energies must limit to 0 as k gets large. It is often useful to know which frequencies have relatively large energies.
A trivial example is the function $\cos(2x)$. The Fourier Series for this function is $\cos(2x)$ (the function is already a member of the basis, there is no need to rewrite it). Thus, the function is equal to its second harmonic, and all of the energy is concentrated in the second harmonic.
For the square wave example in the previous section, the even harmonics are all 0, and harmonic $2k+1$ has energy $\dfrac{16}{n^2}$, so the energies decay quadratically. We can plot these energies in `maple` using the `ColumnGraph` command from the `Statistics` package:

```
with(Statistics):
ColumnGraph([0,16/1,0,16/9,0,16/25,0,16/49],
  width = 0.07, distance = .93);
```

Visually, this graph shows that most of the energy of the square wave is contained in the first harmonic.