1. (Chapter 1: 60 points) Consider the system $A\vec{u} = \vec{b}$ with $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ defined by

$$2x_1 + 3x_2 + 4x_3 + x_4 = 2$$
$$4x_1 + 3x_2 + 8x_3 + x_4 = 4$$
$$6x_1 + 3x_2 + 8x_3 + x_4 = 2$$

$$\Rightarrow \left[\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 2 \\ 4 & 3 & 8 & 1 & 4 \\ 6 & 3 & 8 & 1 & 2 \end{array}\right]$$

Solve the following parts:

A **(a)** [10%] Find the reduced row echelon form of the augmented matrix.

A **(b)** [10%] Identify the **free** variables and the **lead** variables.

A **(c)** [10%] Display a vector formula for a particular solution $\vec{u}_p$. $\Rightarrow \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

A **(d)** [10%] Display a vector formula for the homogeneous solution $\vec{u}_h$. $\Rightarrow t_1 \begin{bmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{bmatrix}$

A **(e)** [10%] Identify each of **Strang's Special Solutions.** $\Rightarrow \begin{bmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{bmatrix}$

A **(f)** [10%] Display the vector general solution $\vec{u}$, using **superposition**. $\Rightarrow \vec{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{bmatrix}$

a) $\left[\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 2 \\ 4 & 3 & 8 & 1 & 4 \\ 6 & 3 & 8 & 1 & 2 \end{array}\right] \begin{array}{l} \\ r_2 - 2r_1 \\ r_3 - 3r_1 \end{array} \left[\begin{array}{cccc|c} 2 & 3 & 4 & 1 & 2 \\ 0 & -3 & 0 & -1 & 0 \\ 0 & -6 & -4 & -2 & -4 \end{array}\right] \begin{array}{l} r_1 + r_2 \\ -\frac{1}{3} r_2 \\ r_3 - 2r_2 \end{array} \left[\begin{array}{cccc|c} 2 & 0 & 4 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & -4 & 0 & -4 \end{array}\right]$

$(r_1 + r_3)/2$

$\frac{1}{4} r_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

b) lead var. - x_1, x_2, x_3 ; free var. - x_4

c) $x_1 = -1$
$x_2 = -\frac{1}{3}x_4 = -\frac{1}{3}t_1$
$x_3 = 1$
$x_4 = t_1$

$\Rightarrow \vec{x}_h = t_1 \begin{bmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{bmatrix}_3$, $\vec{x}_p = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \vec{u}_p$

d) $\vec{u}_h = t_1 \begin{bmatrix} 0 \\ -1/3 \\ 0 \end{bmatrix}$

= t1 [0, −1/3, 0, 1]   (page cut off)

2. (Chapter 2: 40 points)

(a) [10%] Describe for $n \times n$ matrices two different methods for finding the matrix inverse.

(b) [20%] Apply the two methods to find the inverse of the matrix $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$.

(c) [10%] Find the inverse of the transpose of the matrix in part (b).

A

1(a) (1) using $A^{-1} = \dfrac{\mathrm{adj}A}{|A|}$

(2) using $(A|I) \sim (I|A^{-1})$

(b) (1) $|A| = 1 \times 2 = 2$ $\mathrm{adj}A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ $\therefore A^{-1} = \dfrac{\mathrm{adj}}{|A|} = \frac{1}{2}\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

(2) $\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right) \therefore A^{-1} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

(c) $(A^T)^{-1} = (A^{-1})^T = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}$

3. (Chapter 3: 30 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

A

$$x_3 = \frac{\det(A_3(\vec{b}))}{\det A}$$

$$A_3(\vec{b}) = \begin{pmatrix} -2 & 3 & 1 \\ 0 & -2 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\det(A_3(\vec{b})) = 1 \times \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} + 3 \times \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \times (6+2) + 3 \times 4 = 8 + 12 = 20$$

$$\det A = 1 \times \begin{vmatrix} 3 & 0 \\ -2 & 4 \end{vmatrix} + (-2) \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \times 12 + (-2) \times 4$$

$$= 12 - 8 = 4$$

$$\therefore x_3 = \frac{20}{4} = 5$$

~~4.~~ **(Chapters 1 to 4: 30 points)** Let

Removed from the final exam. Corrections are change R^4 to to R^5 and text "of the matrix" to "of some matrix"

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -3 & -2 & -1 \\ -1 & 0 & 0 \\ 6 & 6 & 3 \\ 2 & 2 & 1 \end{pmatrix}$$

(a) Check the independence tests below which apply to prove that the column vectors of the matrix A are independent in the vector space $\mathcal{R}^4$.

(b) Show the details for one of the independence tests that you checked.

[ ]	**Wronskian test**	Wronskian of $\vec{f}_1, \vec{f}_2, \vec{f}_3$ nonzero at $x = x_0$ implies independence of $\vec{f}_1, \vec{f}_2, \vec{f}_3$.
[x]	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
[ ]	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
[ ]	**Euler Atom test**	Any finite set of distinct atoms is independent.
[ ]	**Sample test**	Functions $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are independent if a sampling matrix has nonzero determinant.
[x]	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
[ ]	**Orthogonality test**	A set of nonzero pairwise orthogonal vectors is independent.
[x]	**Combination test**	A list of vectors is independent if each vector is not a linear combination of the preceding vectors.

Solution: Find the RREF of A. It has 2 pivots. No test applies to prove the columns are independent, because the columns of A are dependent.

5. **(Chapters 2, 4: 20 points)** Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = x_2$, $x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

the vector space for all vectors $\vec{x}$ in R^3 that satisfies three equations are

A

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ -x_1 + x_2 + x_3 = 0 \\ x_3 = 0 \end{cases}$$

so S is null space of the 3×3 system on the left

Thus S is subspace of R^3

Also possible: Apply the Kernel Theorem, which says that a system of linear homogeneous algebraic equations has solution set which is a subspace.

6. (Chapter 6: 40 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}.$$

Find a Gram-Schmidt orthonormal basis of S.

A

Let $\vec{w}_1 = \vec{v}_1$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \vec{v}_1}{\vec{v}_1 \vec{v}_1}\vec{v}_1$$

$$= \begin{pmatrix}1\\1\\0\\1\end{pmatrix} - \frac{1+1+0+0}{1+1+1+0}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\0\\1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}1-\frac{2}{3}\\1-\frac{2}{3}\\0-\frac{2}{3}\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{2}{3}\\1\end{pmatrix}$$

unitize $\vec{w}_1$ $\vec{w}_2$ to get $\vec{u}_1$ $\vec{u}_2$

$$\vec{u}_1 = \frac{\vec{w}_1}{|\vec{w}_1|} = \frac{1}{\sqrt{1+1+1}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\0\end{pmatrix}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{|\vec{w}_2|} = \frac{1}{\sqrt{(\frac{1}{3})^2+(\frac{1}{3})^2+(-\frac{2}{3})^2+1}}\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{2}{3}\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{4}{9}+\frac{9}{9}}}\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{2}{3}\\1\end{pmatrix}$$

$$= \frac{3}{\sqrt{15}}\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\-\frac{2}{3}\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{15}}\\\frac{1}{\sqrt{15}}\\\frac{-2}{\sqrt{15}}\\\frac{3}{\sqrt{15}}\end{pmatrix}$$

$\vec{u}_1$ $\vec{u}_2$ are orthonormal basis for S

7. (Chapters 1 to 6: 30 points) Let A be an $m \times n$ matrix and assume that $A^T A$ has nonzero determinant. Prove that the rank of A equals n.

A

Since A^TA has non zero determinant, A^TA is invertible, ie $(A^TA)^{-1}$ exist

Let $\vec{x}$ be a vector which satisfies $A\vec{x} = \vec{0}$ (1)

(Since A is $m \times n$ $\vec{x}$ would be $n \times 1$)

left multiply by A^T $A^TA\vec{x} = \vec{0}$ (2)

since A^TA has inverse $(A^TA)^{-1}$ left multiply (2) by $(A^TA)^{-1}$

$(A^TA)^{-1} A^TA\vec{x} = 0 \Rightarrow \vec{x} = 0$

Which means the $\vec{x}$ that satisfies $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$

This means $A\vec{x} = \vec{0}$ has only trivial solution, ie all columns in A are pivot columns and $m \geq n$.

Ie there are n pivot columns and thus rank of A is n.

used rank + nullity = # vars = n

rank = n $\Leftrightarrow$ nullity = 0 $\Leftrightarrow$ Null(A) = $\{\vec{0}\}$

8. **(Chapter 5: 40 points)** The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display three eigenpairs of A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

A

Try $(A-3I)\vec{x}=0$ and see how many eigenvectors we find

$$\begin{pmatrix} 4-3 & 1 & 1 \\ -1 & 2-3 & 1 \\ 0 & 0 & 3-3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since there is only one free variable in matrix on the left

It has only one eigenvector.

So A has only one eigenpair, A is not diagonalizable.

9. (Chapter 6: 30 points) Let W be the column space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Let $\hat{\vec{\mathbf{b}}}$ be the near point to $\vec{\mathbf{b}}$ in the subspace W. Find $\hat{\vec{\mathbf{b}}}$.

A

$$A^TA\hat{\vec{x}} = A^T\vec{b}$$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+1+1 & 1+1 \\ 1+1 & 1+1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^T\vec{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+(-1)+1 \\ 1+(-1)+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So solve $\hat{\vec{x}}$ by $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ augumented matrix $\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 2 & 2 & 0 \end{array}\right)$

row redue augmented matrix $\left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -1 \end{array}\right)$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right) \therefore \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\hat{\vec{b}} = A\hat{\vec{x}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

10. (Chapter 6: 30 points) Let Q be an orthogonal matrix with columns $\vec{q}_1, \vec{q}_2, \vec{q}_3$. Let D be a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \lambda_3$. Prove that the 3×3 matrix $A = QDQ^T$ satisfies $A = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$.

A

Since $A = QDQ^T$ Q has orthogonal columns $\vec{q}_1\ \vec{q}_2\ \vec{q}_3$ we know $\vec{q}_1\ \vec{q}_2\ \vec{q}_3$ are eigen vector of A
D be diagonal matrix with entries $\lambda_1, \lambda_2 \lambda_3$

so $A\vec{q}_1 = \lambda_1 \vec{q}_1$

multiply $\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$ by $\vec{q}_1$ since $\vec{q}_1 \perp \vec{q}_2$, $\vec{q}_1 \perp \vec{q}_3$

$\vec{q}_1 (\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$

$= \lambda_1 \vec{q}_1$ ($\vec{q}_1$ is unit vector since it came from orthogonal mati

in the same way we can get

$A\vec{q}_2 = \lambda_2 \vec{q}_2 = \vec{q}_2(\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T +)$

$A\vec{q}_3 = \lambda_3 \vec{q}_3 = \vec{q}(\lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3$

so $A = \lambda_1 \vec{q}_1 \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}$

Another proof: $Q = \langle \vec{q}_1 | \vec{q}_2 | \vec{q}_3 \rangle$ $Q^T = \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$

$$A = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \vec{q}_1 & \lambda_2 \vec{q}_2 & \lambda_3 \vec{q}_3 \end{bmatrix} \begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vec{q}_3^T \end{bmatrix}$$

$$= \lambda \vec{q}_1 \cdot \vec{q}_1^T + \lambda_2 \vec{q}_2 \vec{q}_2^T + \lambda_3 \vec{q}_3 \vec{q}_3^T$$

11. (Chapter 7: 30 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

A

find eigenvalues of A

$$\begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$(4-\lambda)(4-\lambda)-(-1)(-1)=0$

$(\lambda-4)(\lambda-4)-1=0$

$\lambda^2-8\lambda+16-1=0$

$\lambda^2-8\lambda+15=0$

$(\lambda-3)(\lambda-5)=0$

$\therefore \lambda_1=5, \lambda_2=3.$

① solve for eigenvector

$$\begin{pmatrix} 4-5 & -1 \\ -1 & 4-5 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \therefore \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

unitize $\vec{v}_1$ get

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

② solve for the second eigenvector

$$\begin{pmatrix} 4-3 & -1 \\ -1 & 4-3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$\therefore \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ unitize $\vec{v}$ get

$$\vec{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

So $D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$ $Q = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ $Q^T = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

12. (Chapter 7: 30 points) Write out the singular value decomposition for the matrix $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$.

$$A^TA = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4+1 & 4-1 \\ 4-1 & 4+1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

Try solve for eigen values of A^TA $\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$

$$(5-\lambda)(5-\lambda) - 9 = 0$$
$$(\lambda-5)(\lambda-5) - 9 = 0$$
$$\lambda^2 - 10\lambda + 25 - 9 = 0$$
$$\lambda^2 - 10\lambda + 16 = 0$$
$$(\lambda-2)(\lambda-8) = 0$$
$$\therefore \lambda_1 = 8 \quad \lambda_2 = 2$$

So singular values for A are

$\sigma_1 = \sqrt{8} = 2\sqrt{2}$

$\sigma_2 = \sqrt{2}$

Solve for eigen vectors of A^TA

$$\begin{pmatrix} 5-8 & 3 \\ 3 & 5-8 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$$

unitize to get $\vec{v}_1$

So $\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Solve for the other eigen vector

$$\begin{pmatrix} 5-2 & 3 \\ 3 & 5-2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

unitize

$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{\sigma_1}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2\sqrt{2}} \cdot \frac{4}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\therefore$ SVD for A is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

13. (Chapter 4: 30 points) Let the linear transformation T from $\mathcal{R}^3$ to $\mathcal{R}^3$ be defined by its action on three independent vectors:

$$T\left(\begin{pmatrix}3\\2\\0\end{pmatrix}\right)=\begin{pmatrix}4\\4\\2\end{pmatrix}, T\left(\begin{pmatrix}0\\2\\1\end{pmatrix}\right)=\begin{pmatrix}5\\1\\1\end{pmatrix}, T\left(\begin{pmatrix}1\\2\\1\end{pmatrix}\right)=\begin{pmatrix}4\\0\\2\end{pmatrix}.$$

A

Find the unique 3×3 matrix A such that T is defined by the matrix multiply equation $T(\vec{x})=A\vec{x}$.

write the three transformatiss into one, ie T work on each column of the matrix on right of dot product form

$$A\begin{pmatrix}3&0&1\\2&2&2\\0&1&1\end{pmatrix}=\begin{pmatrix}4&5&4\\4&1&0\\2&1&2\end{pmatrix}$$

Let B represent $\begin{pmatrix}3&0&1\\2&2&2\\0&1&1\end{pmatrix}$

Since determinant of B is not 0 $|B| = 1\times(-1)\times 4 + 1\times 1\times 6 = 2$

$\therefore B^{-1}$ exist $A=\begin{pmatrix}4&5&4\\4&1&0\\2&1&2\end{pmatrix}\times B^{-1}$

Solve for B^{-1} first

$$\left(\begin{array}{ccc|ccc}3&0&1&1&0&0\\2&2&2&0&1&0\\0&1&1&0&0&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&\frac{1}{3}&\frac{1}{3}&0&0\\1&1&1&0&\frac{1}{2}&0\\0&1&1&0&0&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&\frac{1}{3}&\frac{1}{3}&0&0\\0&1&\frac{2}{3}&-\frac{1}{3}&\frac{1}{2}&0\\0&1&1&0&0&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&\frac{1}{3}&\frac{1}{3}&0&0\\0&1&\frac{2}{3}&-\frac{1}{3}&\frac{1}{2}&0\\0&0&\frac{1}{3}&\frac{1}{3}&-\frac{1}{2}&1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&0&0&\frac{1}{2}&-1\\0&1&0&-1&\frac{3}{2}&-2\\0&0&1&1&-\frac{3}{2}&3\end{array}\right)\therefore B^{-1}=\begin{pmatrix}0&\frac{1}{2}&-1\\-1&\frac{3}{2}&-2\\1&-\frac{3}{2}&3\end{pmatrix}$$

$$A=\begin{pmatrix}4&5&4\\4&1&0\\2&1&2\end{pmatrix}\begin{pmatrix}0&\frac{1}{2}&-1\\-1&\frac{3}{2}&-2\\1&-\frac{3}{2}&3\end{pmatrix}=\begin{pmatrix}0\times4+5\times(-1)+4\times1 & \frac{1}{2}\times4+\frac{3}{2}\times5+\left(\frac{-3}{2}\right)\times4 & 4\times(-1)+5(\ +3\times4\\ 0\times4+1\times(-1)+0 & \frac{1}{2}\times4+\frac{3}{2}\times1 & 4\times(-1)-2\\ 2\times0+1\times(-1)+2\times1 & \frac{1}{2}\times2+\frac{3}{2}\times1-\frac{3}{2}\times2 & -2-2+6\end{pmatrix}$$

Checked with maple. The answer given here is correct.

$$=\begin{pmatrix}-1&\frac{7}{2}&-2\\-1&\frac{7}{2}&-6\\1&-\frac{1}{2}&2\end{pmatrix}$$

14. (Chapter 4, 7: 40 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of of A and S_2 the column space of A. It is known that S_1 and S_2 have dimension $r = \text{rank}(A)$. Let $\vec{p}_1, \ldots, \vec{p}_r$ be a basis for S_1 and let $\vec{q}_1, \ldots, \vec{q}_r$ be a basis for S_2. For example, select the pivot columns of A^T and A, respectively. Define $T : S_1 \to S_2$ initially by $T(\vec{p}_i) = \vec{q}_i, \quad i = 1, \ldots, r$. Extend T to all of S_1 by linearity, which means the final definition is

A or A-

$$T(c_1\vec{p}_1 + \cdots + c_r\vec{p}_r) = c_1\vec{q}_1 + \cdots + c_r\vec{q}_r.$$

one to one:

Prove that T is one-to-one and onto.

Let $\vec{x}$ be a vector in S_1, ie row space of A

Let $\vec{x}_1$ $\vec{x}_2$ be two vectors that are equal after linear tranformatis T, and $\vec{x} = \vec{x}_1$

ie $A\vec{x}_1 = A\vec{x}_2 \Rightarrow A\vec{x}_1 - A\vec{x}_2 = 0$ $A(\vec{x}_1 - \vec{x}_2) = 0$ $A\vec{x} = \vec{0}$

ie $\vec{x}$ in nullspace of A since nullspace of A $\perp$ row-space of A and intersection

so $\vec{x} = \vec{0}$ ie $\vec{x}_1 = \vec{x}_2$ so one to one proven

onto: Let $\vec{y}$ be any vector in S_2, ie column space of A, ie $\vec{y}$ is linear combination of columns of A so there exist a $\vec{x}$ whose entries are weight of linear combination that $A\vec{x} = \vec{y}$ so onto proven

15. (Chapter 4: 20 points) Least squares can be used to find the best fit line for the points $(1,2)$, $(2,2)$, $(3,0)$. Without finding the line equation, describe how to do it, in a few sentences.

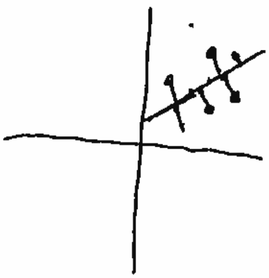
find $\vec{x}$ of $A\vec{x}=\vec{b}$ by, using $y = v_1 x + v_2$, where $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

A

Plugging that into the normal equation $A^TA\vec{y} = A^T\vec{b}$, then solve.

~~The~~ The regression fits a best fit line by taking the average distance from the data points and plots a linear or non-linear line/curve. The best fit line is interpolated from the data points that have been collected.

Picture



$$y = x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

16. (Chapters 1 to 7: 20 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

A

Given matrix A, there are four spaces of A ColA. NulA rowspace of A.
Nul A^T

part two: Nul A $\perp$ to row space of A

Substitute A with A^T in above

Nul $A^T \perp$ ColA

Let r be rank of A. A be mxn

part 1 row space of A has dimension r

$\dim(\text{Col} A) = r$

$\text{Nul} A = n-r$

$\text{Nul} A^T = m-r$

17. (Chapter 7: 20 points) State the **Spectral Theorem** for symmetric matrices. Include the important results included in the spectral theorem, about real eigenvalues and diagonalizability. Then discuss the **spectral decomposition**.

A^{-} or A

Eigenvalues are sometimes called ~~Spectral~~ Spectrum

(1) Eigen vectors of symmetric matrix that correspond to distinct eigen values are orthogonal with each other

(2) dimension of eigen spaces of symmetric matrix are the multiplicity of corresponding eigen values. Also eigenspace that correspond to distinct eigen values are orthogonal to each other.

(3) symmetric matrices are all diagonalizable

(4) If all eigen values are >0 the A is positive definite

If all eigenvalues are <0 then A is negative definite

If A has both positive and negative eigen values, then A is indefinite

(5) D is diagonal with entries coming from A's eigen values

P is orthogonal with columns being A's eigen vector after being unized

then $A = PDP^T = PDP^{-1}$

yes, but Gram-Schmidt is used here also.

An eigenspace may have multiple vectors found as Strang's Special solutions which are not orthogonal. Gram-Schmidt is applied to these vectors to replace the eigenspace basis by an orthogonal basis.