MATH 2270-2 Final Exam Sample Problems Spring 2018

ANSWERS

No books, notes or electronic devices, please.
The questions have credits which reflect the time required to write the solution.
If you must write a solution out of order or on the back side, then supply a road map.
Solutions are expected to include readable and convincing details. A correct answer without details earns 25%.
Only a small sampling from these problem types will appear on the final exam, due to time limitations. Expect about 10 minutes per problem. A final exam problem will have multiple parts.
In addition to these problems, please review the problems from Exam 1 and Exam 2. Sources are located in the CALENDAR at the course web site, week 15.

Problem 1. (5 points) Let A be a 2×2 matrix such that $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$. Compute $A\begin{pmatrix}2\\2\end{pmatrix}$.

Answer:

$$A\left(\begin{array}{c}2\\2\end{array}\right) = 2A\left(\begin{array}{c}1\\1\end{array}\right) = 2\left(\begin{array}{c}1\\0\end{array}\right) = \left(\begin{array}{c}2\\0\end{array}\right)$$

Problem 2. (5 points) State (1) the definition of norm, (2) the Cauchy-Schwartz inequality and (3) the triangle inequality, for vectors in $\mathcal{R}^n$.

Answer:

(1) Norm of $\vec{v}$ equals $\|\vec{v}\| = \sqrt{\vec{v}^T\vec{v}}$; (2) $|\vec{a}\cdot\vec{b}| \le \|\vec{a}\|\|\vec{b}\|$; (3) $\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$.

Problem 3. (5 points) Suppose $A = B(C+D)E$ and all the matrices are $n \times n$ invertible. Find an equation for C.

Answer:

$AE^{-1} = BC + BD$ implies $C = B^{-1}(AE^{-1} - BD)$.

Problem 4. (5 points) Find all solutions to the system of equations

$$\begin{aligned} 2w + 3x + 4y + 5z &= 1 \\ 4w + 3x + 8y + 5z &= 2 \\ 6w + 3x + 8y + 5z &= 1 \end{aligned}$$

Answer:

Infinite solution case: $w = -1/2, x = -(5/3)t_1, y = 1/2, z = t_1$.

Problem 5. (5 points) Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Show the details of two different methods for finding the inverse of the matrix A.

Answer:

The two methods are (1) $A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$ and (2) For $C =< A|I >$, then $\mathbf{rref}(C) =< I|A^{-1} >$. Details expected, but not supplied here.

Problem 6. (5 points) Find a factorization $A = LU$ into lower and upper triangular matrices for the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Answer:

Let E_1 be the result of combo(1,2,-1/2) on I, and E_2 the result of combo(2,3,-2/3) on I. Then $E_2 E_1 A = U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$. Let $L = E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$.

Problem 7. (5 points) Let Q be a 2×2 matrix with $QQ^T = I$. Prove that Q has columns of unit length and its two columns are orthogonal.

Answer:

First, $AB = I$ with both A, B square implies $BA = I$. So $Q^T Q = I$. Then $Q =< \vec{q}_1 | \vec{q}_2 >$ implies $Q^T Q = \begin{pmatrix} \vec{q}_1 \cdot \vec{q}_1 & \vec{q}_1 \cdot \vec{q}_2 \\ \vec{q}_2 \cdot \vec{q}_1 & \vec{q}_2 \cdot \vec{q}_2 \end{pmatrix}$. Relation $Q^T Q = I$ then implies orthogonality of the columns and that the columns have length one.

Problem 8. (5 points) True or False? If the 3×3 matrices A and B are triangular, then AB is triangular.

Answer:

False. Consider the decomposition $A = LU$ in a problem above. True if both matrices are upper triangular or both matrices are lower triangular.

Problem 9. (5 points) True or False? If a 3×3 matrix A has an inverse, then for all vectors $\vec{b}$ the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$.

Answer:

True, $\vec{x} = A^{-1}\vec{b}$.

Problem 10. (5 points) Let A be a 3×4 matrix. Find the elimination matrix E which under left multiplication against A performs both (1) and (2) with one matrix multiply.

(1) Replace Row 2 of A with Row 2 minus Row 1.

(2) Replace Row 3 of A by Row 3 minus 5 times Row 2.

Answer:

Perform combo(1,2,-1) on I then combo(2,3,-5) on the result. The elimination matrix is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & -5 & 1 \end{pmatrix}$$

Problem 11. (10 points) Determinant problem, chapter 3.
(a) [10%] True or False? The value of a determinant is the product of the diagonal elements.
(b) [10%] True or False? The determinant of the negative of the $n \times n$ identity matrix is -1.
(c) [30%] Assume given 3×3 matrices A, B. Suppose $E_2E_1A^2 = AB$ and E_1, E_2 are elementary matrices representing respectively a combination and a multiply by 3. Assume $\det(B) = 27$. Let $C = -A$. Find all possible values of $\det(C)$.
(d) [20%] Determine all values of x for which $(2I + C)^{-1}$ fails to exist, where I is the 3×3 identity and $C = \begin{pmatrix} 2 & x & -1 \\ 3x & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

(e) [30%] Let symbols a, b, c denote constants and define

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & \frac{1}{2} \end{pmatrix}$$

Apply the adjugate [adjoint] formula for the inverse

$$A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$$

to find the value of the entry in row 4, column 2 of A^{-1}.

Answer:

(a) FALSE. True only if the matrix is triangular.
(b) FALSE. It equals 1 when n is even.
(c) Start with the determinant product theorem $|FG| = |F||G|$. Apply it to obtain $|E_2||E_1||A|^2 = |A||B|$. Let $x = |A|$ in this equation and solve for x. You will need to know that $|E_1| = 1$ and $|E_2| = 3$. Then $|C| = |(-I)A| = |-I||A| = (-1)^3 x$. The answer is $|C| = 0$ or $|C| = -9$.
(d) Find $C + I = \begin{pmatrix} 4 & x & -1 \\ 3\,x & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, then evaluate its determinant, to eventually solve for $x = -5/3$ and $x = 2$. Used here is F^{-1} exists if and only if $|F| \neq 0$.
(e) Find the cross-out determinant in row 2, column 4 (no mistake, the transpose swaps rows and columns). Form the fraction, top=checkerboard sign times cross-out determinant, bottom=$|A|$. The value is $-b - a$. A maple check:

```
C4:=Matrix([[1,-1,0,0],[1,0,0,0],[a,b,0,1],[1,c,1,1/2]]);
1/C4; # The inverse matrix
C5:=linalg[minor](C4,2,4);
(-1)**(2+4)*linalg[det](C5)/linalg[det](C4);
# ans = -b-a
```

Problem 12. (5 points) Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 0 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_3 by Cramer's Rule, showing **all details** (details count 75%). To save time, **do not compute** x_1, x_2!

Answer:

$x_3 = \Delta_3/\Delta$, $\Delta = \det \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 1 \\ 1 & 4 & 1 \end{pmatrix} = -8$, $\Delta_3 = \det \begin{pmatrix} -2 & 3 & -3 \\ 0 & -4 & 5 \\ 1 & 4 & 1 \end{pmatrix} = 59$, $x_3 = -\frac{59}{8}$.

Problem 13. (5 points) Define matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

Answer:

Let E_1 be the result of combo(1,2,-1/2) on I, and E_2 the result of combo(2,3,-2/3) on I. Then $E_2E_1A = U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$. Let $L = E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$.

Problem 14. (5 points) Determine which values of k correspond to a unique solution for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-2 & k-3 \\ 1 & 4 & 3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

Answer:

There is a unique solution for $\det(A) \neq 0$, which implies $k \neq 2$ and $k \neq 3$. Alternative solution: Elimination methods with swap, combo, multiply give $\begin{pmatrix} 1 & 4 & k & 1 \\ 0 & k-2 & 0 & k-2 \\ 0 & 0 & 3-k & k-1 \end{pmatrix}$. Then (1) Unique solution for three lead variables, equivalent to the determinant nonzero for the frame above, or $(k-2)(3-k) \neq 0$; (2) No solution for $k = 3$ [signal equation]; (3) Infinitely many solutions for $k = 2$.

Problem 15. (10 points) Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & -b & c \\ 1 & c & a \\ 2 & -b+c & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ -a \\ -a \end{pmatrix}$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a). The system has a unique solution for $(b+c)(2a+c) \neq 0$.

(b). The system has no solution if $2a+c = 0$ and $a \neq 0$ (don't explain the other possibilities).

(c). The system has infinitely many solutions if $a = c = 0$ (don't explain the other possibilities).

Answer:

Combo, swap and mult are used to obtain in 3 combo steps the matrix

$$A_3 = \begin{pmatrix} 1 & -b & c & a \\ 0 & b+c & -c+a & -2a \\ 0 & 0 & -c-2a & -a \end{pmatrix}$$

(a) Uniqueness requires zeros free variables. Then the diagonal entries of the last frame must be nonzero, written simply as $-(c+b)(2a+c) \neq 0$, which is equivalent to the determinant of A not equal to zero.

(b) No solution: The last row of A_3 is a signal equation if $c + 2a = 0$ and $a \neq 0$.

(c) Infinitely many solutions: If $a = c = 0$, then A_3 has last row zero. If $a = c = 0$ and $b = 0$, then there is one lead variable and two free variables, because the last two rows of A_3 are zero. If $a = c = 0$ and $b \neq 0$, then there are two lead variables and one free variable. The homogeneous problem has infinitely many solutions, because of at least one free variable and no signal equation.

The sequence of steps are documented below for maple.

```
with(LinearAlgebra):
combo:=(A,s,t,m)->LinearAlgebra[RowOperation](A,[t,s],m);
```

```
mult:=(A,t,m)->LinearAlgebra[RowOperation](A,t,m);
swap:=(A,s,t)->LinearAlgebra[RowOperation](A,[s,t]);
A:=(a,b,c)->Matrix([[1,b,c,-a],[1,c,-a,a],[2,b+c,a,a]]);
A0:=A(a,b,c);
A1:=combo(A(a,b,c),1,2,-1);
A2:=combo(A1,1,3,-2);
A3:=combo(A2,2,3,-1);
A4:=convert(A3,list,nested=true);
A4 := [[1, -b, c, a], [0, b+c, -c+a, -2*a], [0, 0, -c-2*a, -a]];
```

Problem 16. (5 points) Explain how the **span theorem** applies to show that the set S of all linear combinations of the functions $\cosh x, \sinh x$ is a subspace of the vector space V of all continuous functions on $-\infty < x < \infty$.

Answer:

The span theorem says $\textbf{span}(\vec{v}_1, \vec{v}_2)$ is a subspace of V, for any two vectors in V. Choose the two vectors to be $\cosh x, \sinh x$.

Problem 17. (5 points) Write a proof that the subset S of all solutions $\vec{x}$ in $\mathcal{R}^n$ to a homogeneous matrix equation $A\vec{x} = \vec{0}$ is a subspace of $\mathcal{R}^n$. This is called the **kernel theorem**.

Answer:

(1) Zero is in S because $A\vec{0} = \vec{0}$; (2) If $A\vec{v}_1 = \vec{0}$ and $A\vec{v}_2 = \vec{0}$, then $\vec{v} = \vec{v}_1 + \vec{v}_2$ satisfies $A\vec{v} = A\vec{v}_1 + A\vec{v}_2 = \vec{0} + \vec{0} = \vec{0}$. So $\vec{v}$ is in S; (3) Let $\vec{v}_1$ be in S, that is, $A\vec{v}_1 = \vec{0}$. Let c be a constant. Define $\vec{v} = c\vec{v}_1$. Then $A\vec{v} = A(c\vec{v}_1) = cA\vec{v}_1 = (c)\vec{0} = \vec{0}$. Then $\vec{v}$ is in S. This completes the proof.

Problem 18. (5 points) Using the subspace criterion, write two hypotheses that imply that a set S in a vector space V is not a subspace of V. The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

Answer:

(1) If the zero vector is not in S, then S is not a subspace. (2) If two vectors in S fail to have their sum in S, then S is not a subspace. (3) If a vector is in S but its negative is not, then S is not a subspace.

Problem 19. (5 points) Report which columns of A are pivot columns: $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Answer:

Zero cannot be a pivot column (no leading one in $\mathbf{rref}(A)$). The other two columns are not constant multiples of one another, therefore they are independent and will become pivot columns in $\mathbf{rref}(A)$. Then: pivot columns =2,3.

Problem 20. (5 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Answer:

The augmented matrix has reduced row echelon form (last frame) equal to the matrix $\begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$. Then $x_1 = t_1, x_2 = 1, x_3 = 1$ is the general solution in scalar form. The partial derivative on t_1 gives the homogeneous solution basis vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $\vec{x}_h = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Set $t_1 = 0$ in the scalar solution to find a particular solution $\vec{x}_p = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Problem 21. (5 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

Answer:

The augmented matrix for this system of equations is

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 3 & 0 & 1 & 0 & 4 \\ 4 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The reduced row echelon form is found as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 4 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{combo(1,2,-3)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 0 & 0 & 0 & -16 & 0 \end{pmatrix} \quad \text{combo(1,3,-4)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{mult(3,-1/16)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{last frame}$$

The last frame, or RREF, implies the system

$$\begin{array}{rcl} x_1 & = & 0 \\ x_3 & = & 4 \\ x_4 & = & 0 \end{array}$$

The lead variables are x_1, x_3, x_4 and the free variable is x_2. The last frame algorithm introduces invented symbol t_1. The free variable is set to this symbol, then back-substitute into the lead variable equations of the last frame to obtain the general solution

$$\begin{array}{rcl} x_1 & = & 0, \\ x_2 & = & t_1, \\ x_3 & = & 4, \\ x_4 & = & 0. \end{array}$$

Strang's *special solution* $\vec{s}_1$ is the partial of $\vec{x}$ on the invented symbol t_1. A particular solution $\vec{x}_p$ is obtained by setting all invented symbols to zero. Then

$$\vec{x} = \vec{x}_p + t_1\vec{s}_1 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Problem 22. (5 points) Find the reduced row echelon form of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

Answer:

It is the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Problem 23. (5 points) A 10×13 matrix A is given and the homogeneous system $A\vec{x} = \vec{0}$ is transformed to reduced row echelon form. There are 7 lead variables. How many free variables?

Answer:

Because $\vec{x}$ has 13 variables, then the rank plus the nullity is 13. There are 6 free variables.

Problem 24. (5 points) The rank of a 10×13 matrix A is 7. Find the nullity of A.

Answer:

There are 13 variables. The rank plus the nullity is 13. The nullity is 6.

Problem 25. (5 points) Given a basis $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ of $\mathcal{R}^2$, and $\vec{v} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$, then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ for a unique set of coefficients c_1, c_2, called the *coordinates of* $\vec{v}$ *relative to the basis* $\vec{v}_1, \vec{v}_2$. Compute c_1 and c_2.

Answer:

The question reduces to solving for $\mathbf{x}$ in an equation $A\mathbf{x} = \mathbf{b}$. The entries of $\mathbf{x}$ are the answers c_1, c_2. Matrix $A = \langle \mathbf{v}_1 | \mathbf{v}_2 \rangle$. Vector $\mathbf{b}$ has components $10, 4$ The answer is $\mathbf{x} = A^{-1}\mathbf{b}$, which implies $c_1 = 6, c_2 = -2$.

Problem 26. (5 points) Determine independence or dependence for the list of vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Answer:

Possible tests are the rank test, determinant test, pivot theorem, orthogonality test. Let A denote the augmented matrix of the three column vectors. The determinant is 32, nonzero, so the vectors are independent. The pivot theorem also applies. The $\mathbf{rref}(A)$ is the identity matrix, so all columns are pivot columns, hence the three columns are independent. The rank test applies because the rank is 3, equal to the number of columns, hence independence.

Problem 27. **(5 points)** Check the independence tests which apply to prove that 1, x^2, x^3 are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of $\vec{f}_1, \vec{f}_2, \vec{f}_3$ nonzero at $x = x_0$ implies independence of $\vec{f}_1, \vec{f}_2, \vec{f}_3$.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Euler Atom test**	Any finite set of distinct atoms is independent.
☐	**Sample test**	Functions $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are independent if a sampling matrix has nonzero determinant.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
☐	**Orthogonality test**	A set of nonzero pairwise orthogonal vectors is independent.

Answer:

The first, fourth and fifth apply to the given functions, while the others apply only to fixed vectors.

Problem 28. **(5 points)** Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

Answer:

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then the restriction equations can be written as $A\vec{x} = \vec{0}$. Apply the nullspace theorem (also called the kernel theorem), which says that the nullspace of a matrix is a subspace.

Another solution: The given restriction equations are linear homogeneous algebraic equations. Therefore, S is the nullspace of some matrix B, hence a subspace of $\mathcal{R}^3$. This solution uses the fact that linear homogeneous algebraic equations can be written as a matrix equation $B\vec{x} = \vec{0}$.

Another solution: Verify the three checkpoints for a subspace S in the Subspace Criterion. This is quite long, and certainly the last choice for a method of proof.

Problem 29. (5 points) The 5×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 6 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Answer:

Find $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The pivot columns are 1 and 4.

Problem 30. (5 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Find a Gram-Schmidt orthonormal basis of S.

Answer:

Let $\vec{y}_1 = \vec{v}_1$ and $\vec{u}_1 = \frac{1}{\|\vec{y}_1\|}\vec{y}_1$. Then $\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Let $\vec{y}_2 = \vec{v}_2$ minus the shadow projection of $\vec{v}_2$ onto the span of $\vec{v}_1$. Then

$$\vec{y}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

Finally, $\vec{u}_2 = \frac{1}{\|\vec{y}_2\|}\vec{y}_2$. We report the Gram-Schmidt basis:

$$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix}.$$

Problem 31. (5 points) Find the orthogonal projection vector $\vec{v}$ (the shadow projection vector) of $\vec{v}_2$ onto $\vec{v}_1$, given

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Answer:

Use the formula $\vec{v} = d\vec{v}_1$ where $d = \dfrac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}$.

Problem 32. (5 points) Let A be an $m \times n$ matrix with independent columns. Prove that $A^T A$ is invertible.

Answer:

The matrix $B = A^T A$ has dimension $n \times n$. We prove that the nullspace of $B = A^T A$ is the zero vector.
Let $\vec{x}$ belong to $\mathcal{R}^n$. Assume $B\vec{x} = \vec{0}$, then multiply this equation by $\vec{x}^T$ to obtain $\vec{x}^T A^T A\vec{x} = \vec{x}^T\vec{0} = 0$. Therefore, $\|A\vec{x}\|^2 = 0$, or $A\vec{x} = \vec{0}$. If A has independent columns, then the nullspace of A is the zero vector, so $\vec{x} = \vec{0}$. We have proved that the nullspace of $B = A^T A$ is the zero vector.
An $n \times n$ matrix B is invertible if and only if its nullspace is the zero vector. So $B = A^T A$ is invertible.

Problem 33. (5 points) Let A be an $m \times n$ matrix with $A^T A$ invertible. Prove that the columns of A are independent.

Answer:

The columns of A are independent if and only if the nullspace of A is the zero vector. If you don't know this result, then find it in Strang's book, or prove it yourself.
Assume $\vec{x}$ is in the nullspace of A, $A\vec{x} = \vec{0}$, then multiply by A^T to get $A^T A\vec{x} = \vec{0}$. Because $A^T A$ is invertible, then $\vec{x} = \vec{0}$, which proves the nullspace of A is the zero vector. We conclude that the columns of A are independent.

Problem 34. (5 points) Let A be an $m \times n$ matrix and $\vec{v}$ a vector orthogonal to the nullspace of A. Prove that $\vec{v}$ must be in the row space of A.

Answer:

The fundamental theorem of linear algebra is summarized by `rowspace` $\perp$ `nullspace`. This relation implies `nullspace` $\perp$ `rowspace`, because for subspaces S we have $(S^\perp)^\perp = S$. The conclusion follows.

Problem 35. (5 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_2 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

Answer:

$$x_2 = \Delta_2/\Delta,\ \Delta_2 = \det \begin{pmatrix} -2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 3 & -2 \end{pmatrix} = 36,\ \Delta = \det(A) = 4,\ x_2 = 9.$$

Problem 36. (5 points) Assume $A^{-1} = \begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix}$. Find the inverse of the transpose of A.

Answer:

Compute $(A^T)^{-1} = (A^{-1})^T = \left(\begin{pmatrix} 2 & -6 \\ 0 & 4 \end{pmatrix} \right)^T = \begin{pmatrix} 2 & 0 \\ -6 & 4 \end{pmatrix}$.

Problem 37. (5 points) This problem uses the identity $A\,\mathbf{adj}(A) = \mathbf{adj}(A)A = |A|I$, where $|A|$ is the determinant of matrix A. Symbol $\mathbf{adj}(A)$ is the adjugate or adjoint of A. The identity is used to derive the adjugate inverse identity $A^{-1} = \mathbf{adj}(A)/|A|$..

Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix is $C = \mathbf{adj}(B)$. Report the value of the determinant of matrix $C^{-1}B^2$.

$$B = \begin{pmatrix} 1 & -1 & ? & ? \\ 1 & ? & 0 & 0 \\ ? & 0 & 2 & ? \\ ? & 0 & 0 & ? \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 & 2 & 0 \\ -4 & 4 & -2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Answer:

The determinant of $C^{-1}B^2$ is $|B|^2/|C|$. Then $CB == \mathbf{adj}(B)B = |B|I$ implies $|C||B| = \det(|B|I) = |B|^4$. Because $|C| = |B|^3$, then the answer is $1/|B|$. Return to $CB = |B|I$ and do one dot product to find the value $|B| = 8$. We report $\det(C^{-1}B^2) = 1/|B| = 1/8$.

Problem 38. (5 points) Display the entry in row 3, column 4 of the adjugate matrix [or adjoint matrix] of $A = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. Report both the symbolic formula and the numerical value.

Answer:

The answer is the cofactor of A in row 4, column 3 = $(-1)^7$ times minor of A in 4,3 = -2.

Problem 39. (5 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer:

The columns of P are the eigenvectors and the diagonal entries of D are the eigenvalues, taken in the same order.

Problem 40. (5 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer:

The characteristic polynomial is $\det(A - rI) = (-r)(3-r)(r-2)^2$. The eigenvalues are $0, 2, 2, 3$. Determinant expansion of $\det(A - \lambda I)$ is by the cofactor method along column 1. This reduces it to a 3×3 determinant, which can be expanded by the cofactor method along column 3.

Problem 41. (5 points) The matrix $A = \begin{pmatrix} 0 & -12 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $0, 2, 2$ but it is not diagonalizable, because $\lambda = 2$ has only one eigenpair. Find an eigenvector for $\lambda = 2$. To save time, **don't find the eigenvector for** $\lambda = 0$.

Answer:

Because $A - 2I = \begin{pmatrix} -2 & -12 & 3 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ has last frame $B = \begin{pmatrix} 1 & 0 & -15/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then there is only one eigenpair for $\lambda = 2$, with eigenvector $\vec{v} = \begin{pmatrix} 15 \\ -2 \\ 2 \end{pmatrix}$.

Problem 42. (5 points) Find the two complex eigenvectors corresponding to complex eigenvalues $-1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

Answer:

$$\left(-1 + 2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(-1 - 2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$$

Problem 43. (5 points) Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Answer:

The first and the last, because the test $A\vec{x} = \lambda\vec{x}$ passes in both cases.

Problem 44. (5 points) Let I denote the 3×3 identity matrix. Assume given two 3×3 matrices B, C, which satisfy $CP = PB$ for some invertible matrix P. Let C have eigenvalues -1, 1, 5. Find the eigenvalues of $A = 2I + 3B$.

Answer:

Both B and C have the same eigenvalues, because $\det(B - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(PCP^{-1} - \lambda PP^{-1}) = \det(C - \lambda I)$. Further, both B and C are diagonalizable. The answer is the same for all such matrices, so the computation can be done for a diagonal matrix $B = \mathbf{diag}(-1, 1, 5)$. In this case, $A = 2I + 3B = \mathbf{diag}(2, 2, 2) + \mathbf{diag}(-3, 3, 15) = \mathbf{diag}(-1, 5, 17)$ and the eigenvalues of A are $-1, 5, 17$.

Problem 45. (5 points) Let A be a 3×3 matrix with eigenpairs

$$(4, \vec{v}_1), \quad (3, \vec{v}_2), \quad (1, \vec{v}_3).$$

Let P denote the augmented matrix of the eigenvectors $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_1$, in exactly that order. Display the answer for $P^{-1}AP$. Justify the answer with a sentence.

Answer:

Because $AP = PD$, then $D = P^{-1}AP$ is the diagonal matrix of eigenvalues, taken in the order determined by the eigenpairs $(3, \vec{v}_2), (1, \vec{v}_3), (4, \vec{v}_1)$. Then $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Problem 46. (5 points) The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display three eigenpairs of A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Answer:

Compute $\mathbf{rref}(A-3I) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This has rank 2, nullity 1. There is just one eigenvector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ for $\lambda = 3$. Not diagonalizable, no Fourier's model, not possible to display three eigenpairs.

Problem 47. (5 points) Assume A is a given 4×4 matrix with eigenvalues $0, 1, 3 \pm 2i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix.

Answer:

Such a matrix is diagonalizable, because of four distinct eigenvalues. Then $4B - 3I$ has the same eigenvalues for all matrices B similar to A. In particular, $4A - 3I$ has the same eigenvalues as $4D-3I$ where D is the diagonal matrix with entries $0, 1, 3+2i, 3-2i$. Compute $4D - 3I = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9+8i & 0 \\ 0 & 0 & 0 & 9-8i \end{pmatrix}$. The answer is $0, 1, 9+8i, 9-8i$.

Problem 48. (5 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer:

The characteristic polynomial is $\det(A - rI) = (r^2+6)(3-r)(r-2)^2$. The eigenvalues are $2, 2, 3, \pm\sqrt{6}i$. Determinant expansion is by the cofactor method along column 5. This reduces it to a 4×4 determinant, which can be expanded as a product of two quadratics. In detail, we first get $|A - rI| = (3-r)|B - rI|$, where $B = \begin{pmatrix} 0 & -2 & -5 & 0 \\ 3 & 0 & -12 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$. So we have one eigenvalue 3, and we find the eigenvalues of B. Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer for the eigenvalues of A is 3 and the eigenvalues of B_1 and B_3. We report $3, \pm\sqrt{6}i, 2, 2$. It is also possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$.

Problem 49. (5 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(3, \begin{pmatrix} 13 \\ 6 \\ -41 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

(1) [10%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [10%] Display a matrix product formula for A, but do not evaluate the matrix products, in order to save time.

Answer:

(1) $P = \begin{pmatrix} 13 & i & -i \\ 6 & 2 & 2 \\ -41 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}$. (2) $AP = PD$ implies $A = PDP^{-1}$.

Problem 50. (5 points) Assume two 3×3 matrices A, B have exactly the same characteristic equations. Let A have eigenvalues 2, 3, 4. Find the eigenvalues of $(1/3)B - 2I$, where I is the identity matrix.

Answer:

Because the answer is the same for all matrices similar to A (that is, all $B = PAP^{-1}$) then it suffices to answer the question for diagonal matrices. We know A is diagonalizable, because it has distinct eigenvalues. So we choose D equal to the diagonal matrix with entries $2, 3, 4$. Compute $\frac{1}{3}D - 2I = \begin{pmatrix} \frac{2}{3} - 2 & 0 & 0 \\ 0 & \frac{3}{3} - 2 & 0 \\ 0 & 0 & \frac{4}{3} - 2 \end{pmatrix}$. Then the eigenvalues are $-\frac{4}{3}, -1, -\frac{2}{3}$.

Problem 51. (5 points) Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove that the roots of the characteristic equations of A and B are identical.

Answer:

The proof depends on the identity $A - rI = PBP^{-1} - rI = P(B - rI)P^{-1}$ and the determinant product theorem $|CD| = |C||D|$. We get $|A - rI| = |P||B - rI||P^{-1}| = |PP^{-1}||B - rI| = |B - rI|$. Then A and B have exactly the same characteristic equation, hence exactly the same eigenvalues.

Problem 52. (5 points) Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Answer:

The characteristic polynomial is $\det(B - rI) = (2 - r)(5 - r)(5 - r)(3 - r)$. The eigenvalues are $2, 3, 5, 5$.

It is possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$.

An alternate method is described below, which depends upon a determinant product theorem for special block matrices, such as encountered in this example.

Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer is that B has eigenvalues equal to the eigenvalues of B_1 and B_3. These are quickly found by Sarrus' Rule applied to the two 2×2 determinants $|B_1 - rI| = (2 - r)(5 - r)$ and $|B_3 - rI| = r^2 - 8r + 15 = (5 - r)(3 - r)$.

Problem 53. (5 points) Let W be the column space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let

$\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Let $\hat{\vec{\mathbf{b}}}$ be the near point to $\vec{\mathbf{b}}$ in the subspace W. Find $\hat{\vec{\mathbf{b}}}$.

Answer:

The columns of A are independent. The normal equation is $A^TA\vec{y} = A^T\vec{b}$, which in explicit form is $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}\vec{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The answer is $\vec{y} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $\hat{\vec{\mathbf{b}}} = \mathbf{A\tilde{y}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Problem 54. (5 points) There are real 2×2 matrices A such that $A^2 = -4I$, where I is the identity matrix. Give an example of one such matrix A and then verify that $A^2 + 4I = 0$.

Answer:

Choose any matrix whose characteristic equation is $\lambda^2 + 4 = 0$. Then $A^2 + 4I = 0$ by the Cayley-Hamilton theorem.

Problem 55. (5 points) Let $Q = <\vec{q}_1|\vec{q}_2>$ be orthogonal 2×2 and D a diagonal matrix with diagonal entries λ_1, λ_2. Prove that the 2×2 matrix $A = QDQ^T$ satisfies $A = \lambda_1\vec{q}_1\vec{q}_1^T + \lambda_2\vec{q}_2\vec{q}_2^T$.

Answer:

Let $B = \lambda_1\vec{q}_1\vec{q}_1^T + \lambda_2\vec{q}_2\vec{q}_2^T$. We prove $A = B$. First observe that both A and B are symmetric. Because the columns of Q form a basis of $\mathcal{R}^2$, it suffices to prove that $\vec{x}^TA = \vec{x}^TB$ for $\vec{x}$ a column of A. For example, take $\vec{x} = \vec{q}_1$. Then $\vec{x}^TA = (A^T\vec{q}_1)^T = (A\vec{q}_1)^T = \lambda_1\vec{q}_1^T$. Orthogonality of Q implies $\vec{x}^TB = (B\vec{q}_1)^T = (\lambda_1\vec{q}_1\vec{q}_1^T\vec{q}_1 + \lambda_2\vec{q}_2\vec{q}_2^T\vec{q}_1)^T = \lambda_1(\vec{q}_1 \cdot 1)^T = \lambda_1\vec{q}_1^T$. Repeat for subscript 2 to complete the proof.

Problem 56. (5 points) A matrix A is **defined** to be positive definite if and only if $\vec{x}^TA\vec{x} > 0$ for nonzero $\vec{x}$. Which of these matrices are positive definite?

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 2 & -6 \end{pmatrix}$$

Answer:

Only the second matrix. A useful test is positive eigenvalues. Another is principal determinants all positive.

Problem 57. (5 points) Let A be a real symmetric 2×2 matrix. Prove that the eigenvalues of A are real numbers.

Answer:

Begin with $A\vec{x} = \lambda\vec{x}$. Take the conjugate of both sides to get a new equation. Because the conjugate of a real matrix is itself, then the new equation looks like $A\vec{y} = \bar{\lambda}\vec{y}$ where $\vec{y}$ is the conjugate of $\vec{x}$. Formally, replace i by $-i$ in the components of $\vec{x}$ to obtain $\vec{y}$. Symbol $\bar{\lambda}$ is the complex conjugate of λ. Transpose this new equation to get $\vec{y}^T A = \bar{\lambda}\vec{y}^T$, possible because $A = A^T$. Taking dot products two ways gives $\vec{y} \cdot A\vec{x} = \lambda\vec{y} \cdot \vec{x} = \bar{\lambda}\vec{y} \cdot \vec{x}$. Because $\vec{y} \cdot \vec{x} = \|\vec{x}\|^2 > 0$, then we can cancel to get $\lambda = \bar{\lambda}$, proving the eigenvalue λ is real.

Problem 58. (5 points) Let B be a real 3×4 matrix. Prove that the eigenvalues of $B^T B$ are non-negative.

Answer:

Let $A = B^T B$. An eigenpair $(\lambda, \vec{v})$ of A satisfies $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq \vec{0}$. Already known is that the eigenvalue λ and the eigenvector $\vec{v}$ are real, because $A = B^T B$ is a symmetric matrix. Compute $\|B\vec{v}\|^2 = (B\vec{v})^T(B\vec{v}) = \vec{v}^T B^T B\vec{v} = \vec{v}^T A\vec{v} = \vec{v}^T \lambda\vec{v} = \lambda\|\vec{v}\|^2$. Therefore, λ is non-negative.

Problem 59. (5 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$.

Answer:

Start with the equation $r^2 - 6r + 8 = 0$ having roots $r = 2, 4$. Compute the eigenpairs $(2, \vec{v}_1)$, $(4, \vec{v}_2)$ where $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The two vectors are orthogonal but not of unit length. Unitize them to get $\vec{u}_1 = \frac{1}{\sqrt{2}}\vec{v}_1$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\vec{v}_2$. Then $Q =< \vec{u}_1 | \vec{u}_2 >= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \mathbf{diag}(2, 4)$.

Problem 60. (5 points) Show that if B is an invertible matrix and A is similar to B, with $A = PBP^{-1}$, then A is invertible.

Answer:

The determinant product theorem applies to obtain $|A| = |B| \neq 0$, hence A is invertible.

Problem 61. (5 points) Write out the singular value decomposition for the matrix $A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$.

Answer:

$$A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right)^T$$

Problem 62. (5 points) Strang's Four Fundamental Subspaces are the nullspace of A, the nullspace of A^T, the row space of A and the column space of A. Describe, using a figure or drawing, the locations in the matrices U, V of the singular value decomposition $A = U\Sigma V^T$ which are consumed by the four fundamental subspaces of A.

Answer:

$A = \langle \mathbf{colspace}(A) | \, \mathbf{nullspace}(A^T) \rangle \cdot \Sigma \cdot \langle \mathbf{rowspace}(A) | \, \mathbf{nullspace}(A) \rangle^T$. The dimensions of the spaces left to right are r, $m - r$, r, $n - r$, where A is $m \times n$ and r is the rank of A.

Problem 63. (5 points) Give examples for a vertical shear and a horizontal shear in the plane. Expected is a 2×2 matrix A which represents the linear transformation.

Answer:

$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ is a horizontal shear, $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ is a vertical shear

Problem 64. (5 points) Give examples for clockwise and counterclockwise rotations in the plane. Expected is a 2×2 matrix A which represents the linear transformation.

Answer:

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for $\theta > 0$ rotates clockwise and for $\theta < 0$ rotates counter clockwise.

Problem 65. (5 points) Let the linear transformation T from $\mathcal{R}^3$ to $\mathcal{R}^3$ be defined by its action on three independent vectors:

$$T\left(\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}, T\left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

Find the unique 3×3 matrix A such that T is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$.

Answer:

$A \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 5 \\ 4 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ can be solved for matrix A. The answer is $A = \begin{pmatrix} 1 & \frac{1}{2} & 3 \\ 1 & \frac{1}{2} & -1 \\ -1 & \frac{5}{2} & -3 \end{pmatrix}$.

Problem 66. (5 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of A and S_2 the column space of A. Prove that $T : S_1 \to S_2$ defined by $T(\vec{x}) = A\vec{x}$ is one-to-one and onto.

Answer:

Suppose $\vec{x}$ is in the rowspace. The fundamental theorem of linear algebra says $\vec{x}$ is perpendicular to the nullspace. of A. So, if $\vec{x}_1, \vec{x}_2$ are vectors in the rowspace of A and $A\vec{x}_1) = A(\vec{x}_2$ then $A(\vec{x}_1 - \vec{x}_2) = \vec{0}$. This implies $\vec{x} = \vec{x}_1 - \vec{x}_2$ belongs to the nullspace of A. But $\vec{x}$ is a linear combination of vectors in S_1, so it is in S_1, which is perpendicular to the nullspace. The intersection of V and $V^\perp$ is the zero vector, so $\vec{x} = \vec{0}$, which says $\vec{x}_1 = \vec{x}_2$, proving T is one-to-one.

The proof for onto is done by solving the equation $A\vec{x} = \vec{y}$ where $\vec{y}$ is any vector in the column space of A. We have to find $\vec{x}$ in S_1 that solves the equation. Select any $\vec{z}$ such that $\vec{y} = A\vec{z}$. Because the rowspace is perpendicular to the nullspace, then there are unique vectors $\vec{x}, \vec{u}$ such that $\vec{z} = \vec{x} + \vec{u}$, and $\vec{u}$ is in the nullspace while $\vec{x}$ is in the rowspace. Then $\vec{y} = A\vec{z} = A\vec{x} + A\vec{u} = A\vec{x} + \vec{0} = A\vec{x}$. We have solved the equation for $\vec{x}$ in S_1. The proof is complete.

Essay Questions

Problem 67. (5 points) Define an Elementary Matrix. Display the fundamental matrix multiply equation which summarizes a sequence of swap, combo, multiply operations, transforming a matrix A into a matrix B.

Answer:

An elementary matrix is the matrix E resulting from one elementary row operation (swap, combination, multiply) performed on the identity matrix I. The fundamental equation looks like $E_k \cdots E_2 E_1 A = B$, but this is not the complete answer, because the elementary matrices have to be explained, relative to the elementary row operations which transformed A into B.

Problem 68. (5 points) Let V be a vector space and S a subset of V. Define what it means for S to be a **subspace** of V. The definition is sometimes called the **Subspace Criterion**, a theorem with three requirements, with the conclusion that S is a subspace of V.

Answer:

The definition can be found in the textbook, although the naming convention might not be the same. In some books it is taken as the definition, in other books it is derived from a different definition, then recorded as a theorem called the subspace criterion: (1) Zero is in S; (2) Sums of vectors in S are in S; (3) Scalar multiples of vectors in S are in S. The important underlying assumption is that addition and scalar multiplication are inherited from V.

Problem 69. (5 points) The null space S of an $m \times n$ matrix M is a subspace of $\mathbb{R}^n$. This is called the *Kernel Theorem*, and it is proved from the **Subspace Criterion**. Both theorems conclude that some subset is a subspace, but they have different hypotheses. Distinguish the Kernel theorem from the Subspace Criterion, as viewed from hypotheses.

Answer:

The distinction is that the kernel theorem applies only to fixed vectors, that is, the vector space $\mathcal{R}^n$, whereas the subspace criterion applies to any vector space.

Problem 70. (5 points) Least squares can be used to find the best fit line for the points $(1, 2)$, $(2, 2)$, $(3, 0)$. Without finding the line equation, describe how to do it, in a few sentences.

Answer:

Find a matrix equation $A\vec{x} = \vec{b}$ using the line equation $y = v_1 x + v_2$ where $\vec{x} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then solve the normal equation $A^T A\vec{v} = A^T\vec{b}$. A full solution is expected, with a formula for A. But don't solve the normal equation.

Problem 71. (5 points) State the Fundamental Theorem of Linear Algebra. Include

Part 1: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

Answer:

Part 1. The dimensions are $n - r, r, rm - r$ for nullspace(A), colspace(A), rowspace(A), nullspace(A^T). Part 2. The orthogonality relation is rowspace $\perp$ nullspace, for both A and A^T. A full statement is expected, not the brief one given here.

Problem 72. **(5 points)** Display the equation for the Singular Value Decomposition (SVD), then cite the conditions for each matrix. Finish with a written description of how to construct the matrices in the SVD.

Answer:

Let r be the rank of an $m \times n$ matrix A. Then $A = U\Sigma V^T$, where $A\vec{v}_i = \sigma_i \vec{u}_i$, $U =< \vec{u}_1 | \cdots | \vec{u}_n >$, $V =< \vec{v}_1 | \cdots | \vec{v}_m >$ are orthogonal and $\Sigma = \mathbf{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0 >$. The singular values are $\sigma_i = \sqrt{\lambda_i}$ where $\{\lambda_i\}$ is the list of real nonnegative eigenvalues of $A^T A$. Only the positive values σ_i, $i = 1, \ldots, r$ where r is the rank of A are entered into matrix Σ, and they must be ordered in decreasing order. Because there is a full set of n orthonormal eigenpairs $(\lambda, \vec{v})$ for the $n \times n$ symmetric matrix $A^T A$, the matrix V is constructed from the list of orthonormal eigenvectors $\{\vec{v}_i \|_{i=1}^n$. Matrix U is constructed from an orthonormal basis $\{\vec{u}_i\}_{i=1}^m$, obtained from Gram-Schmidt, starting with the list of orthogonal vectors $\vec{u}_i = \frac{}{\sigma_i} A\vec{v}_i$, $i = 1, \ldots, r$, after appending to the list $m - r$ independent vectors to complete a basis of $\mathcal{R}^m$.

Problem 73. **(5 points)** State the **Spectral Theorem** for symmetric matrices. Include the important results included in the spectral theorem, about real eigenvalues and diagonalizability. Then discuss the **spectral decomposition**.

Answer:

A real symmetric $n \times n$ matrix A has only real eigenvalues. Matrix A has n eigenpairs, in short it is diagonalizable. To each eigenvalue of multiplicity k, there are k independent eigenvectors. These eigenvectors span a subspace of dimension k which by Gram-Schmidt is spanned by k orthonormal vectors. Two such subspaces corresponding to different eigenvalues are orthogonal.
The spectral decomposition of A is $A = QDQ^T$ where D is a diagonal matrix of eigenvalues and Q is an orthogonal matrix of corresponding eigenvectors.

Problem 74. **(5 points)** State the **Perron-Frobenius Theorem** for positive stochastic matrices A. Include the results for eigenvalue one and the limit of A^k as $k \to \infty$. This is the theorem used as a basis for the *Google Search Algorithm*.

Answer:

The eigenspace of $\lambda = 1$ is one-dimensional: there is a vector $\mathbf{w} \neq \mathbf{0}$ such that $A\mathbf{w} = \mathbf{w}$ and any other solution of $A\mathbf{v} = \mathbf{v}$ is a scalar multiple of $\mathbf{w}$. All other eigenvalues λ of A satisfy $|\lambda| < 1$. The eigenvector $\mathbf{w}$ can be selected such that $\lim_{k\to\infty} A^k = \langle \mathbf{w}| \cdots |\mathbf{w}\rangle$.