Math 2270-2 Final Exam Solutions Spring 2017
17 Questions, Total = 540

1. (Chapter 1: 50 points) Consider the system $A\vec{u} = \vec{b}$ with $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ defined by

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + x_4 + x_5 &= 2 \\ 2x_1 + x_2 + 8x_3 + x_4 + 2x_5 &= 4 \\ 2x_1 + 2x_2 + 8x_3 + x_4 + x_5 &= 2 \end{aligned}$$

Solve the following parts (a) to (e):

(a) [10%] Find the reduced row echelon form of the augmented matrix.

$$\left[\begin{array}{ccccc|c} 1 & 3 & 4 & 1 & 1 & 2 \\ 2 & 1 & 8 & 1 & 2 & 4 \\ 2 & 2 & 8 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 1 & 1 & 2 \\ 0 & -5 & 0 & -1 & 0 & 0 \\ 0 & -4 & 0 & -1 & -1 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1/5 & 0 & 0 \\ 0 & 1 & 0 & 1/4 & 1/4 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 4 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 5 & 10 \end{array}\right] = \text{RREF}$$

(b) [10%] Write the scalar equations corresponding to the answer in (a). Then identify the free variables and the lead variables.

$$\begin{aligned} x_1 + 4x_3 - x_5 &= -2 \\ x_2 - x_5 &= -2 \\ x_4 + x_5 &= 10 \end{aligned}$$

Lead: x_1, x_2, x_4

Free: x_3, x_5

(c) [10%] Display a formula for the **vector general solution** $\vec{u}$.

$$\vec{u} = \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} t_2 + \begin{pmatrix} -2 \\ -2 \\ 0 \\ 10 \\ 0 \end{pmatrix}$$

(d) [10%] Extract from the answer in (c) vector formulas for a particular solution $\vec{u}_p$ and the homogeneous solution $\vec{u}_h$.

$$\vec{u}_p = \begin{pmatrix} -2 \\ -2 \\ 0 \\ 10 \\ 0 \end{pmatrix} \qquad \vec{u}_h = \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} t_1 + \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} t_2$$

(e) [10%] Extract from the answer in (d) a vector solution basis for $A\vec{u} = \vec{0}$. These vectors are called **Strang's Special Solutions**.

$$\begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

2. (Chapter 2: 40 points)

Define $A = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}$ and $B = A + A^T$, where A^T is the transpose of A.

(a) [20%] Apply two different methods to find the inverse of the matrix A.

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \frac{1}{5-4}\begin{pmatrix} 5 & -4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -1 & 1 \end{pmatrix}$$

$$(A \mid I) \sim (I \mid A^{-1}) \rightarrow \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 5 & -4 \\ 0 & 1 & -1 & 1 \end{array}\right) \rightarrow A^{-1} = \begin{pmatrix} 5 & -4 \\ -1 & 1 \end{pmatrix}$$

(b) [20%] Compute $(B^{-1})^T$.

$$(B^{-1})^T = (B^T)^{-1}$$

$$B^T = (A + A^T)^T = \left(\begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}\right)^T = \begin{pmatrix} 2 & 5 \\ 5 & 10 \end{pmatrix}^T = \begin{pmatrix} 2 & 5 \\ 5 & 10 \end{pmatrix}$$

$$(B^T \mid I) \sim (I \mid (B^T)^{-1}) \rightarrow \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 5 & 10 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 0 & 5 & 5 & -2 \\ 1 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 5 & 5 & -2 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2/5 \end{array}\right)$$

$$(B^T)^{-1} = \boxed{\begin{pmatrix} -2 & 1 \\ 1 & -2/5 \end{pmatrix} = (B^{-1})^T}$$

3. (Chapter 3: 30 points) Let P, Q, R be real numbers. Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

Find the value of x_2 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

$$x_2 = \frac{\begin{vmatrix} -2 & P & 0 \\ 0 & Q & 4 \\ 1 & R & -2 \end{vmatrix}}{\begin{vmatrix} -2 & 2 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -2 \end{vmatrix}} = \frac{4Q + 8R + 4P}{4} = \boxed{Q + 2R + P}$$

$$\begin{vmatrix} -2 & P & 0 \\ 0 & Q & 4 \\ 1 & R & -2 \end{vmatrix} = -2\begin{vmatrix} Q & 4 \\ R & -2 \end{vmatrix} + 1\begin{vmatrix} P & 0 \\ Q & 4 \end{vmatrix} = -2(-2Q - 4R) + (4P - 0)$$

$$= 4Q + 8R + 4P$$

$$\begin{vmatrix} -2 & 2 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -2 \end{vmatrix} = -2\begin{vmatrix} -1 & 4 \\ 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} = -2(2 - 0) + (8 - 0)$$

$$= -4 + 8 = 4$$

4. (Chapters 1 to 4: 20 points) It is known that functions x, $\cos(x)$, e^x are independent in the vector space V of all functions on $(-\infty, \infty)$. Define functions in V by the equations $f_1(x) = x + e^x$, $f_2(x) = 2x - e^x$, $f_3(x) = 3\cos(x) + x + e^x$.

Definition: *An Euler solution atom is a base atom multiplied by a factor $x^n e^{ax}$ where $n = 0, 1, 2, \ldots$ and a is a real constant. A base atom is one of 1, $\cos(bx)$, $\sin(bx)$ where $b > 0$ is real.*

Check the independence tests below which apply to prove that the functions f_1, f_2, f_3 are independent in the vector space V. Don't check one which won't work!

☑ **Wronskian test** — Wronskian of $\vec{f_1}, \vec{f_2}, \vec{f_3}$ nonzero at $x = x_0$ implies independence of $\vec{f_1}, \vec{f_2}, \vec{f_3}$.

☐ **Euler Solution Atom test** — Any finite set of distinct Euler atoms is independent.

☑ **Sample test** — Functions $\vec{f_1}, \vec{f_2}, \vec{f_3}$ are independent if a sampling matrix has nonzero determinant.

5. (Chapters 1 to 4: 30 points) It is known that functions x, $\cos(x)$, e^x are independent in the vector space V of all functions on $(-\infty, \infty)$. Define functions in V by the equations $f_1(x) = x + e^x$, $f_2(x) = 2x - e^x$, $f_3(x) = 3\cos(x) + x + e^x$.

(a) [10%] Independence of the functions f_1, f_2, f_3 in the vector space V can be established using the coordinate map

$$c_1 x + c_2 e^x + c_3 \cos(x) \quad \text{maps into} \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Reformulate the independence of functions f_1, f_2, f_3 into independence of column vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in the vector space $\mathcal{R}^3$.

$f_1 \to \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad f_2 \to \vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ $\quad f_3 \to \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

(b) [10%] Show details for **one** of the tests below applied to $\vec{v}_1, \vec{v}_2, \vec{v}_3$, defined in part (a).

Pivot Test: $\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

There are 3 pivot columns, therefore the functions are independent (coordinate maps map independent sets to independent sets).

(c) [10%] Check all tests below that may be applied to $\vec{v}_1, \vec{v}_2, \vec{v}_3$, as defined in part (a). Don't check a test which won't work!

- [x] **Rank test** — Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
- [x] **Determinant test** — Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
- [x] **Pivot test** — Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
- [ ] **Orthogonality test** — A set of nonzero pairwise orthogonal vectors is independent.
- [x] **Combination test** — A list of vectors is independent if each vector is not a linear combination of the preceding vectors.

6. (Chapters 2, 4: 20 points) Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + 2x_3 - x_2 = 0$, $x_3 = 0$ and $x_3 + x_2 = x_1$. Supply the proof details which verify that S is a subspace of $\mathcal{R}^3$.

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ formed from $\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ -x_1 + x_2 + x_3 &= 0 \\ x_3 &= 0 \end{aligned}$

Then $A\vec{x} = \vec{0}$ shows that the equations are in the nullspace of R^3. By the nullspace theorem, which states that any nullspace forms a subspace, S is a subspace of R^3. The kernel theorem also states that any linear homogenous equations form a subspace.

7. (Chapter 6: 40 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{x}_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}.$$

(a) [10%] Explain, by citing a theorem, why S is a subspace.

By the subspace criterion: the zero vector is in S, S is closed under vector addition (that is, for two vectors in S their sum is in S), and S is closed under scalar multiplication (that is, if a vector is in S, any scalar times that vector is in S).

(b) [30%] Find a Gram-Schmidt orthonormal basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$ for subspace S.

$$\vec{V}_1 = \vec{X}_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} \qquad \vec{V}_2 = \vec{X}_2 - \frac{\vec{X}_2 \cdot \vec{V}_1}{\vec{V}_1 \cdot \vec{V}_1}\vec{V}_1 = \begin{pmatrix}1\\-1\\0\\1\end{pmatrix} - \frac{0}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}$$

$$\vec{V}_3 = \vec{X}_3 - \frac{\vec{X}_3 \cdot \vec{V}_2}{\vec{V}_2 \cdot \vec{V}_2}\vec{V}_2 - \frac{\vec{X}_3 \cdot \vec{V}_1}{\vec{V}_1 \cdot \vec{V}_1}\vec{V}_1 = \begin{pmatrix}1\\1\\0\\1\end{pmatrix} - \begin{pmatrix}1/3\\-1/3\\0\\1/3\end{pmatrix} - \begin{pmatrix}2/3\\2/3\\2/3\\0\end{pmatrix} = \begin{pmatrix}0\\2/3\\-2/3\\2/3\end{pmatrix}$$

$$\vec{u}_1 = \frac{\vec{V}_1}{\|\vec{V}_1\|} = \frac{1}{\sqrt{3}}\vec{V}_1 = \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\\0\end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{3}}\vec{V}_2 = \begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\0\\1/\sqrt{3}\end{pmatrix} \qquad \vec{u}_3 = \frac{1}{\sqrt{4/9 + 4/9 + 4/9}}\vec{V}_3 = \frac{3}{\sqrt{12}}\begin{pmatrix}0\\2/3\\-2/3\\2/3\end{pmatrix} = \begin{pmatrix}0\\2/\sqrt{12}\\-2/\sqrt{12}\\2/\sqrt{12}\end{pmatrix}$$

8. (Chapters 1 to 6: 30 points) Let A be an $m \times n$ matrix and assume that $A^T A$ has rank $n - 1$. Prove that the rank of A cannot equal n.

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in progress

Let $\vec{x}$ be given such that $A^T A \vec{x} = \vec{0}$

Because $A^T A$ has rank $n-1$, $A^T A$ does not have independent columns, therefore $\vec{x} \neq \vec{0}$.

$$A^T A \vec{x} = \vec{0}$$

Left multiply by $\vec{x}^T$:

$$\vec{x}^T A^T A \vec{x} = \vec{0}$$

$$(A\vec{x})^T A\vec{x} = \vec{0}$$

$$\|A\vec{x}\|^2 = \vec{0}$$

$$A\vec{x} = \vec{0}$$

Because $\vec{x}$ is known to NOT be the zero vector, then the columns are A cannot be independent (because the equation $A\vec{x} = \vec{0}$ does not have only the trivial solution). Therefore the rank of A cannot be n (as if the rank of A was n then A would have independent columns).

9. (Chapter 5: 40 points) The matrix A below has eigenvalues 2, 3 and 3. Compute all eigenpairs of A. Is A diagonalizable?

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A - 2I = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \longrightarrow \left(2, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right) = \text{eigenpair } 1$$

$$A - 3I = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \left(3, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right) = \text{eigenpair } 2$$

A is not diagonizable as it only has 2 eigenpairs. It needs 3 eigenpairs to be diagonizable.

10. (Chapter 6: 30 points) Define $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Let W be the column space of A. Write the normal equations for the inconsistent problem $A\vec{x} = \vec{b}$ and solve for the least squares solution $\hat{\vec{x}}$.

Remark. *Vector $\hat{\vec{\mathbf{b}}} = \mathbf{A}\hat{\vec{\mathbf{x}}}$ is the near point to $\vec{\mathbf{b}}$ in the subspace W.*

Normal equations : $A^T A \hat{\vec{x}} = A^T \vec{b}$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 3 & 0 & 6 \\ 0 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -\frac{1}{2} \end{array}\right) \rightarrow \boxed{\hat{\vec{x}} = \begin{pmatrix} 2 \\ -1/2 \end{pmatrix}}$$

$$\hat{\vec{b}} = A \begin{pmatrix} 2 \\ -1/2 \end{pmatrix}$$

11. (Chapter 6: 30 points) Given vectors $\vec{q}_1, \vec{q}_2, \vec{q}_3$ in $\mathcal{R}^3$, define

$$A = 2\vec{q}_1\vec{q}_1^{\,T} + 5\vec{q}_2\vec{q}_2^{\,T} + 7\vec{q}_3\vec{q}_3^{\,T}.$$

(a) [10%] Prove that A is symmetric.

Symmetric matrices : $A^T = A$

$$A^T = 2(\vec{q}_1\,\vec{q}_1^{\,T})^T + 5(\vec{q}_2\,\vec{q}_2^{\,T})^T + 7(\vec{q}_3\,\vec{q}_3^{\,T})^T$$

$$A^T = 2(\vec{q}_1^{\,T^T}\,\vec{q}_1^{\,T}) + 5(\vec{q}_2^{\,T^T}\,\vec{q}_2^{\,T}) + 7(\vec{q}_3^{\,T^T}\,\vec{q}_3^{\,T})$$

$$A^T = 2\,\vec{q}_1\,\vec{q}_1^{\,T} + 5\,\vec{q}_2\,\vec{q}_2^{\,T} + 7\,\vec{q}_3\,\vec{q}_3^{\,T} = A$$

(b) [20%] The Spectral Theorem for symmetric matrices produces a similar formula where $2, 5, 7$ are replaced by the eigenvalues of A. Write the formula for 3×3 matrices A and explain all the symbols used in the formula.

$$A = Q\,D\,Q^T = \begin{pmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \vec{q}_1 \\ \vec{q}_2 \\ \vec{q}_3 \end{pmatrix}$$

D is a diagonal matrix of eigenvalues.

Q is an orthogonal matrix formed from the normalized eigenvectors of A, using Gram-Schmidt if necessary to obtain orthogonal vectors.

12. (Chapter 7: 30 points) The spectral theorem says that a symmetric matrix A satisfies $AQ = QD$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

$$|A - \lambda I| = (4-\lambda)(4-\lambda) - 1 = \lambda^2 - 8\lambda + 16 - 1$$

$$= \lambda^2 - 8\lambda + 15 = (\lambda - 5)(\lambda - 3)$$

$$\lambda = 5,\ 3$$

$$A - 5I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$A - 3I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \qquad D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}$$

13. (Chapter 7: 40 points) Write out the singular value decomposition for the matrix $A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$.

$$A^T A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 26 & 10 \\ 10 & 26 \end{pmatrix}$$

$$|A^T A - \lambda I| = (26-\lambda)(26-\lambda) - 100$$

$$= \lambda^2 - 52\lambda + 576 - 100$$

$$= \lambda^2 - 52\lambda + 476$$

$$= (\lambda - 36)(\lambda - 16)$$

$$\lambda = 36, 16 \rightarrow \sigma_1 = \sqrt{36} = 6 \qquad \sigma_2 = \sqrt{16} = 4$$

$$A^T A - 36I = \begin{pmatrix} -10 & 10 \\ 10 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$A^T A - 16I = \begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{1}{6}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 6/\sqrt{2} \\ 6/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \frac{1}{4}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -4/\sqrt{2} \\ 4/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

14. (Chapter 4: 30 points) Let the linear transformation T from $\mathcal{R}^2$ to $\mathcal{R}^2$ be defined by its action on two independent vectors:

$$T\left(\begin{pmatrix}3\\2\end{pmatrix}\right)=\begin{pmatrix}4\\2\end{pmatrix},\quad T\left(\begin{pmatrix}2\\1\end{pmatrix}\right)=\begin{pmatrix}5\\1\end{pmatrix}.$$

Find the unique 2×2 matrix A such that T is defined by the matrix multiply equation $T(\vec{x})=A\vec{x}$

$$A\underbrace{\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix}}_{B}=\underbrace{\begin{bmatrix}4 & 5\\ 2 & 1\end{bmatrix}}_{C}$$

$$A = CB^{-1}$$

$$B^{-1}=\frac{1}{\|B\|}\cdot \mathrm{adj}\,B = -1\begin{bmatrix}1 & -2\\ -2 & 3\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 2 & -3\end{bmatrix}$$

$$CB^{-1}=\begin{bmatrix}4 & 5\\ 2 & 1\end{bmatrix}\begin{bmatrix}-1 & 2\\ 2 & -3\end{bmatrix}=\begin{bmatrix}-4+10 & 8-15\\ -2+2 & 4-3\end{bmatrix}=\begin{bmatrix}6 & -7\\ 0 & 1\end{bmatrix}$$

Check

$$\begin{bmatrix}6 & -7\\ 0 & 1\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix}=\begin{bmatrix}18-14\\ 0+2\end{bmatrix}=\begin{bmatrix}4\\ 2\end{bmatrix} \checkmark$$

$$A=\begin{bmatrix}6 & -7\\ 0 & 1\end{bmatrix}$$

15. (Chapter 4, 7: 40 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of of A and S_2 the column space of A. Using only the Pivot Theorem and the Toolkit of swap, combo, multiply, prove that S_1 and S_2 have the same dimension.

By the pivot theorem the # of pivots of A = the rank of A = dimension col (A)

For RREF(A) the # of pivots of A = the # of non-zero rows of A.

The non-zero rows of RREF(A) are the pivots of row-(A) or col (A^T)

# of non-zero rows RREF(A) = # of pivots of A = # of pivots of row(A) = rank row(A) = dim(S_2) = dim(S_1)

16. (Chapter 4: 20 points) Least squares can be used to find the best fit line $y = ax+b$ for the points $(1,2)$, $(2,2)$, $(3,0)$. Find the line equation by the method of least squares.

$$\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}2\\2\\0\end{bmatrix}$$

$$A \qquad \vec{x} = \vec{b}$$

$$A^TA\hat{x} = \hat{b}$$

$$\begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix} = \begin{bmatrix}1&2&3\\1&1&1\end{bmatrix}\begin{bmatrix}2\\2\\0\end{bmatrix}$$

$$\begin{bmatrix}14&6\\6&3\end{bmatrix}\hat{x}=\begin{bmatrix}6\\4\end{bmatrix}$$

$$\left[\begin{array}{cc|c}14&6&6\\6&3&4\end{array}\right]$$

$$\left[\begin{array}{cc|c}1&1/2&2/3\\7&3&3\end{array}\right] c(1,2,-7)$$

$$\left[\begin{array}{cc|c}1&1/2&2/3\\0&-1/2&-5/3\end{array}\right]$$

$$\left[\begin{array}{cc|c}1&1/2&2/3\\0&-1/2&-5/3\end{array}\right] c(2,1,1)$$

$$\left[\begin{array}{cc|c}1&0&-1\\0&-1/2&-5/3\end{array}\right] m(-2)$$

$$\left[\begin{array}{cc|c}1&0&-1\\0&1&10/3\end{array}\right] \qquad \begin{array}{l}a=-1\\b=10/3\end{array}$$

$$y = -x + 10/3$$

17. (Chapters 1 to 7: 20 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

For an $m \times n$ matrix rank A + nul $A = n$

$\dim(\text{row}(A)) = \dim(\text{col}(A)) = n - \dim \text{nul}(A) = n - \dim \text{nul}(A^T)$

$\text{row}(A)^\perp = \text{nul}(A)$

$\text{Col}(A)^\perp = \text{nul}(A^T)$

$\text{Col}(A)$ subspace of $\mathbb{R}^m$

$\text{row}(A)$ subspace of $\mathbb{R}^n$

$\text{nul}(A)$ subspace of $\mathbb{R}^n$

$\text{nul}(A^T)$ subspace of $\mathbb{R}^m$