MATH 2270-2 Exam 2 S2016

ANSWERS

Problem 1. (100 points) Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 0 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_3 by Cramer's Rule, showing **all details** (details count 75%). To save time, **do not compute** x_1, x_2!

Answer:

$$x_3 = \Delta_3/\Delta,\ \Delta = \det\begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 1 \\ 1 & 4 & 1 \end{pmatrix} = 8,\ \Delta_3 = \det\begin{pmatrix} -2 & 3 & -3 \\ 0 & -4 & 5 \\ 1 & 4 & 1 \end{pmatrix} = 51,\ x_3 = \frac{51}{8}.$$

Problem 2. (100 points) Define matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

Answer:

Let E_1 be the result of combo(1,2,-1) on I, and E_2 the result of combo(2,3,-1) on I. Then $E_2E_1A = U = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Let $L = E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

Problem 3. (100 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

Please display answers for both $\vec{x}_h$ and $\vec{x}_p$. The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Answer:

The augmented matrix has reduced row echelon form (last frame) equal to the matrix $\begin{pmatrix} 1 & 0 & -1/6 & 1/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $x_3 = t_1, x_2 = -t_1/2 + 1/2, x_1 = t_1/6 + 1/2$ is the general solution in scalar form. The partial derivative on t_1 gives the homogeneous solution basis vector $\begin{pmatrix} -1/6 \\ -1/2 \\ 1 \end{pmatrix}$. Then $\vec{x}_h = c_1 \begin{pmatrix} -1/6 \\ -1/2 \\ 1 \end{pmatrix}$. Set $t_1 = 0$ in the scalar solution to find a particular solution $\vec{x}_p = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$.

Problem 4. (100 points) Given a basis $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$ for a subspace S of $\mathcal{R}^3$, and $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ in S, then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ for a unique set of coefficients c_1, c_2, called the *coordinates of $\vec{v}$ relative to the basis $\vec{v}_1, \vec{v}_2$*. Compute c_1 and c_2.

Answer:

$c_1 = -1, c_2 = 1$.

Problem 5. (100 points) The functions 1, x^2, $\sqrt{x^7}$ are independent in the vector space V of all functions on $0 < x < \infty$. Check the independence tests which apply.

☐	**Wronskian test**	Wronskian determinant of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Sampling test**	Sampling determinant for samples $x = x_1, x_2, x_3$ nonzero implies independence of f_1, f_2, f_3.
☐	**Rank test**	Three vectors are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Three vectors are independent if their augmented matrix is square and has nonzero determinant.
☐	**Orthogonality test**	Three vectors are independent if they are all nonzero and pairwise orthogonal.
☐	**Pivot test**	Three vectors are independent if their augmented matrix A has 3 pivot columns.

Answer:

The first and second apply to the given functions, while the others apply only to fixed vectors.

Problem 6. (100 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(2, \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}\right), \quad \left(1+i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(1-i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right).$$

(a) [60%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(b) [40%] Display a matrix product formula for A. To save time, **do not evaluate any matrix products**.

Answer:

(a) The columns of P are the eigenvectors and the diagonal entries of D are the eigenvalues $2, 1+i, 1-i$, taken in the same order. (b) The matrix formula comes from $AP = PD$, by solving for $A = PDP^{-1}$.

Problem 7. (100 points) The matrix $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ -1 & -1 & 4 \end{pmatrix}$ has eigenvalues $2, 4, 4$. Find all eigenvectors for $\lambda = 4$. To save time, **don't find the eigenvector for** $\lambda = 2$. Then report whether or not matrix A is diagonalizable.

Answer:

Because $A - 4I = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$ has RREF $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then there are two eigenpairs for $\lambda = 4$, with eigenvectors $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. These answers are Strang's special solutions for the homogeneous problem $B\vec{v} = \vec{0}$. The matrix is diagonalizable, because there are three eigenpairs.

Problem 8. (100 points) Using the subspace criterion, write three different hypotheses each of which imply that a set S in a vector space V is not a subspace of V. The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

Answer:

(1) If the zero vector is not in S, then S is not a subspace.
(2) If two vectors in S fail to have their sum in S, then S is not a subspace.
(3) If a vector is in S but its negative is not, then S is not a subspace.

Problem 9. (100 points) Define S to be the set of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = x_2$, $x_3 + x_2 = x_1$ and $x_1 - x_3 = 0$. Prove that S is a subspace of $\mathcal{R}^3$.

Answer:

Let $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$. Then the restriction equations can be written as $A\vec{x} = \vec{0}$. Apply the nullspace theorem (also called the kernel theorem), which says that the nullspace of a matrix is a subspace.
Another solution: The given restriction equations are linear homogeneous algebraic equations. Therefore, S is the nullspace of some matrix B, hence a subspace of $\mathcal{R}^3$. This solution uses the fact that linear homogeneous algebraic equations can be written as a matrix equation $B\vec{x} = \vec{0}$.

Problem 10. (100 points) Let A be a 100×29 matrix. Assume the columns of A^TA are independent. Prove that A has independent columns.

Answer:

The columns of matrix B are independent if and only if the nullspace of B is the zero vector. If you don't know this result, then find it in Lay's book, or prove it yourself. Also used: A square matrix has independent columns if and only if it is invertible.
Assume $\vec{x}$ is in the nullspace of A, $A\vec{x} = \vec{0}$, then multiply by A^T to get $A^TA\vec{x} = \vec{0}$. Because A^TA is invertible, then $\vec{x} = \vec{0}$, which proves the nullspace of A is the zero vector. We conclude that the columns of A are independent.

Problem 11. (100 points) Let 3×3 matrices A, B and C be related by $AP = PB$ and $BQ = QC$ for some invertible matrices P and Q. Prove that the characteristic equations of A and C are identical.

Answer:

The proof depends on the identity $A - rI = PBP^{-1} - rI = P(B - rI)P^{-1}$ and the determinant product theorem $|CD| = |C||D|$. We get $|A - rI| = |P||B - rI||P^{-1}| = |PP^{-1}||B - rI| = |B - rI|$. Then A and B have exactly the same characteristic equation. Similarly, B and C have the same characteristic equation. Therefore, A and C have the same characteristic equation.

Problem 12. (100 points) The **Fundamental Theorem of Linear Algebra** says that the null space of a matrix is orthogonal to the row space of the matrix.

Let A be an $m \times n$ matrix. Define subspaces S_1 = column space of A, S_2 = null space of A^T. Prove that a vector $\vec{v}$ orthogonal to S_2 must be in S_1.

Answer:

The fundamental theorem of linear algebra is summarized by `rowspace` $\perp$ `nullspace`. Replace A by A^T. Then the result says that `colspace(A)` $\perp$ `nullspace(`A^T`)` or $S_1 \perp S_2$. Results known are dim nullspace = nullity, dim rowspace = rank, and rank + nullity = column dimension of the matrix. Because $S_1 \perp S_2$, then the two subspaces $S_1, S_2 =$ intersect in the zero vector. Their dimensions are **rank**(A) and **nullity**(A^T). Because **rank**(A) = **rank**(A^T), then their two dimensions add to m, as follows: $\dim(S_1) + \dim(S_2) =$ **rank**(A) + **nullity**(A^T) = **rank**(A^T) + **nullity**(A^T) $= m$. Orthogonality of the two subspaces implies a basis for S_1 added to a basis for S_2 supplies a basis for $\mathcal{R}^m$. Vector $\vec{v}$ is given to be orthogonal to S_2. Then it must have a basis expansion involving only the basis for S_1 =`colspace(A)`. This proves that $\vec{v}$ is in S_1, which is the column space of A.
A shorter proof is possible, starting with the result that $\mathcal{R}^n$ equals the direct sum of the subspaces S_1 and S_2. The above details proved this result, as part of the solution. Then the only details left are to add the bases together and expand $\vec{v}$ in terms of the basis elements. The basis elements from S_2 are orthogonal to $\vec{v}$, therefore $\vec{v}$ is expressed as a linear combination of basis elements from S_1, which proves that $\vec{v}$ is in S_1 = the column space of A.