1. (10 points)

(a) Give a counter example or explain why it is true. If A and B are $n \times n$ invertible, and C^T denotes the transpose of a matrix C, then $((A + B)^{-1})^T = (B^T)^{-1} + (A^T)^{-1}$.

(b) Give a counter example or explain why it is true. If A is a square matrix and $A^T A = I$, then both A and A^T are invertible.

2. (10 points) Let A be a 3×4 matrix. Find the elimination matrix E which under left multiplication against A performs both (1) and (2) with one matrix multiply.

(1) Replace Row 2 of A with Row 2 minus Row 1.

(2) Replace Row 3 of A by Row 3 minus 5 times Row 2.

3. (30 points) Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & -b & c \\ 1 & c & a \\ 2 & -b+c & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ -a \\ -a \end{pmatrix}$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a). The system has a unique solution for $(b+c)(2a+c) \neq 0$.

(b). The system has no solution if $2a+c = 0$ and $a \neq 0$ (don't explain the other possibilities).

(c). The system has infinitely many solutions if $a = c = 0$ (don't explain the other possibilities).

4. (20 points) Definition. *Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called* **independent** *provided solving the equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ has the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called* **dependent**.

Find a largest set of independent vectors from the following set of vectors, using the Pivot Theorem or the definition of independence (above).

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

5. (20 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

6. (20 points) Determinant problem, chapter 3. Parts reduced on Exam 1.

(a) [10%] True or False? The value of a determinant is the product of the diagonal elements.

(b) [10%] True or False? The determinant of the negative of the $n \times n$ identity matrix is -1.

(c) [30%] Assume given 3×3 matrices A, B. Suppose $E_2 E_1 A^2 = AB$ and E_1, E_2 are elementary matrices representing respectively a combination and a multiply by 3. Assume $\det(B) = 27$. Let $C = -A$. Find all possible values of $\det(C)$.

(d) [20%] Determine all values of x for which $(2I + C)^{-1}$ fails to exist, where I is the 3×3 identity and $C = \begin{pmatrix} 2 & x & -1 \\ 3x & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

(e) [30%] Let symbols a, b, c denote constants and define

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & \frac{1}{2} \end{pmatrix}$$

Apply the adjugate [adjoint] formula for the inverse

$$A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$$

to find the value of the entry in row 4, column 2 of A^{-1}.

<div align="center">End Exam 1.</div>