Math 310

Applied Linear Algebra

John Lesieutre

MATH 310, APPLIED LINEAR ALGEBRA
FALL 2015 SYLLABUS

COURSE DESCRIPTION: The course will focus on matrix and vector methods for studying systems of linear equations, with an emphasis on concrete calculations and applications. Specific topics to be covered include matrices, Gaussian elimination, vector spaces, *LU*-decomposition, orthogonality, the Gram–Schmidt process, determinants, inner products, eigenvalue problems, and applications to differential equations and Markov processes.

Math 210 is a prerequisite; credit will not be given for both Math 310 and Math 320.

MEETINGS: MWF, 1:00-1:50, Lecture Center D5

TEACHING STAFF: Instructor: John Lesieutre (luh-SEERT)
Email: `jdl@uic.edu`
Office hours: SEO 411, Tu 3-4, Th 10-11, F 3-4
or by appointment (really!)
Online, Tu evening (`math310john@gmail.com` on gchat)

Teaching Assistant: TBD, `TBD`

TEXTBOOK: *Linear Algebra and Its Applications* (5th edition) by David C. Lay, Steven R. Lay, and Judi J. McDonald. (Note: the 4th edition is similar and should be fine for the readings, but you should check homework problems from the book against the 5th.)

WEBSITES: There are three sites for the course:

Course materials: My site (`http://jdl.people.uic.edu/courses/m310f15/`)
Grades: Blackboard (`http://blackboard.uic.edu/`)
Q&A: Piazza (`https://piazza.com/class/idg9ci3gjpq10a`)

IMPORTANT DATES: There will be three exams throughout the semester: two midterms (W 9/30, W 11/11), and the final (M 12/7, 1-3 PM) . The precise topics on the midterms will be announced closer to the dates. Arrangements for make-up exams should be made *before* the date of the exam.

The add/drop deadline is 9/4. The withdraw deadline is 10/30.

HOMEWORK AND QUIZZES: Homework problems will be assigned every week, but will not be graded. Problems on the material for the week will all be posted by Wednesday, and should be completed by the following Wednesday.

You are strongly encouraged to work through all the homework problems and to discuss them with your classmates. Discussions of homework problems are fair game on the Piazza discussion forum.

Every Wednesday there will be a quiz with a problem similar to one from the homework due that day, drawn from material covered during the preceding week. Your ten best quizzes will count for 20% of your final grade.

Grading: The breakdown of the grades is as follows: Quizzes, 20%; Midterm 1, 20%; Midterm 2, 20%; Final, 40%.

Questions: Please ask questions! There are several ways to reach me outside of the lectures, and I'll try to respond quickly:

Piazza:	Q&A site. The best option for math-related questions.
Email:	You can reach me at `jdl@uic.edu`.
Office hours:	Stop by to talk about homework, etc.
Course page:	An anonymous email contact form is available on the course website if you have any questions/suggestions you wouldn't share otherwise. Remember that I won't be able to answer if you don't leave your address.

Attendance: Students are expected to attend every lecture, and should let me know ahead of time if they will miss one. I will occasionally pass around a sign-in sheet during the lecture; students who miss more than four of these will lose 3% from the final grade for each absence.

Other resources:

MSLC:	The Mathematical Sciences Learning Center is located in SEO 430. It offers a place to study and get help with Math 310 throughout the semester. Fall 2015 hours are 8:00 AM to 6:00 PM, M-F.
OCW:	You can find lecture videos from the MIT offering of a similar course at `http://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/`. Coverage of the two courses is broadly similar and some students have found these helpful in the past.

The other sections: A warning: although there are several sections of Math 310 running in parallel, they are not coordinated, and the topics covered, exam dates, etc. won't necessarily coincide (although they are likely to be very similar).

While you are of course encouraged to talk to your friends in the other sections about the material, be sure to check dates, exam coverage etc. for this section only.

Academic integrity: Instances of academic misconduct will be handled according to the Student Disciplinary Policy.

Disability accomodations: Students with disabilities who require accommodation should register with the Disability Resource Center (DRC).

ROUGH CALENDAR

(subject to change)

Week 1		
8/24	§1.1	Linear systems
8/26	§1.2	Row reduction and echelon forms
8/28	§1.3	Vector equations
Week 2		
8/31	§1.4	The matrix equation $A\mathbf{x} = \mathbf{b}$
9/1	§1.5	Solution sets of linear systems
9/3	§1.6	Applications of linear systems
Week 3		
9/9	§1.7	Linear independence
9/11	§1.8	Linear transformations
Week 4		
9/14	§1.9	Matrix of a linear transformation
9/16	§1.10	Linear models in business, science, and engineering
9/18	§2.1	Matrix operations
Week 5		
9/21	§2.2	Inverse of a matrix
9/23	§2.3	Characterization of invertible matrices
9/25	§2.5	Matrix factorization
Week 6		
9/28	§3.1	Introduction to determinants
9/30		Test 1
10/2	§3.2	Properties of determinants
Week 7		
10/5	§3.3	Cramer's rule, volume and linear transformations
10/7	§4.1	Vector spaces and subspaces
10/9	§4.2	Null spaces, column spaces, and linear transformations
Week 8		
10/12	§4.3	Linearly independent sets; bases
10/14	§4.4	Coordinate system
10/16	§4.5	The dimension of a vector space
Week 9		
10/19	§4.6	Rank
10/21	§4.7	Change of basis
10/23	§5.1	Eigenvectors and eigenvalues
Week 10		
10/26	§5.2	The characteristic equation
10/28	§5.3	Diagonalization
10/30	§5.4	Eigenvectors and linear transformations
Week 11		
11/2	Appendix B	Review of complex numbers
11/4	§5.5	Complex eigenvalues
11/6	§4.9	Applications to Markov chains

Week 12		
11/9	§5.7	Applications to differential equations
11/11		Test 2
11/13	§6.1	Inner product, length and orthogonality
Week 13		
11/14	§6.2	Orthogonal sets
11/16	§6.3	Orthogonal projections
11/18	§6.4	The Gram–Schmidt process
Week 14		
11/23	§6.5	Least-squares problems
11/25	§6.6	Application to linear models
Week 15		
11/30	§6.7	Inner product spaces
12/2	§7.1	Diagonalization of symmetric matrices
12/4	§7.4	The singular value decomposition
Week 16		
12/7		Final exam (1-3 PM)

Systems of linear equations:

a bunch of equations of the form

$$a_1x_1 + \cdots + a_nx_n = b$$

<u>example</u>

$$\left.\begin{array}{l} 2x+3y+4z=8 \\ x-y-z=5 \end{array}\right\} \text{– how to find all solutions?}$$

<u>not example</u>

$$2x\underbrace{y-3\sqrt{x}}_{\text{not allowed!}}=7$$

in class, usually 2 or 3 variables.

in real life, could be thousands!

$$-x+y=3$$
$$2x+y=1$$

1) $-x+y=3$

$3y=7 \leftarrow 1+2\cdot 3$

(add 2× eqn 1 to eqn 2)

so $y=7/3$

⇓

$x=-2/3$

The New York Times
SundayReview | The Opinion Pages

OPINION

Is Algebra Necessary?

Adam Hayes

By ANDREW HACKER
Published: July 28, 2012 | 477 Comments

A TYPICAL American school day finds some six million high school students and two million college freshmen struggling with algebra. In both high school and college, all too many students are expected to fail. Why do we subject American students to this ordeal? I've found myself moving toward the strong view that we shouldn't.

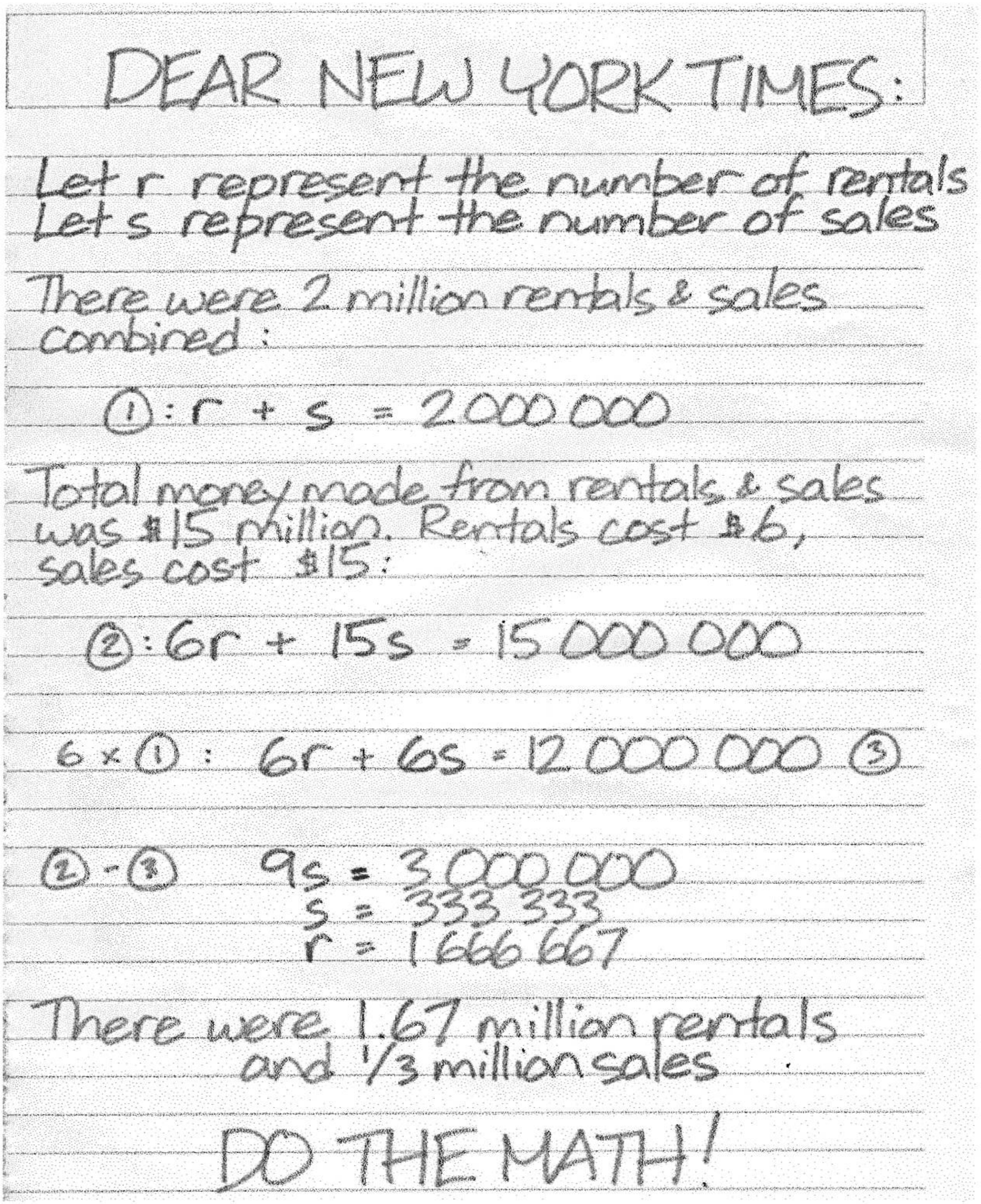
MEDIA

‘The Interview’ Brings In $15 Million on Web

By MICHAEL CIEPLY DEC. 28, 2014

LOS ANGELES — “The Interview” generated roughly $15 million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday.

Sony did not say how much of that total represented $6 digital rentals versus $15 sales. The studio said there were about two million transactions over all.



DEAR NEW YORK TIMES:

Let r represent the number of rentals
Let s represent the number of sales

There were 2 million rentals & sales combined:

① : $r + s = 2\,000\,000$

Total money made from rentals & sales was $15 million. Rentals cost $6, sales cost $15:

② : $6r + 15s = 15\,000\,000$

6 × ① : $6r + 6s = 12\,000\,000$ ③

② - ③ $9s = 3\,000\,000$
$s = 333\,333$
$r = 1\,666\,667$

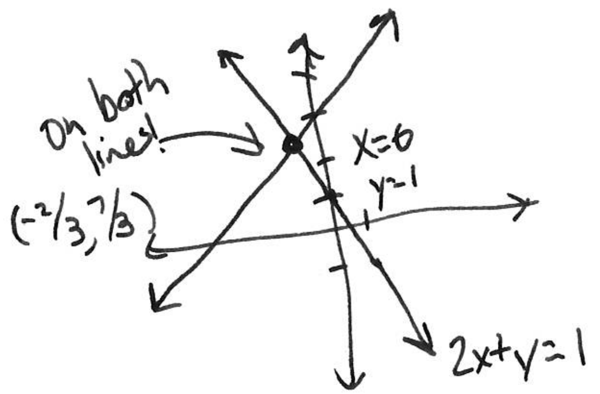
There were 1.67 million rentals and 1/3 million sales

DO THE MATH!

2 eqns, 2 vars.

$$2x+y=1$$
$$-x+y=3$$

both of these are equations of line



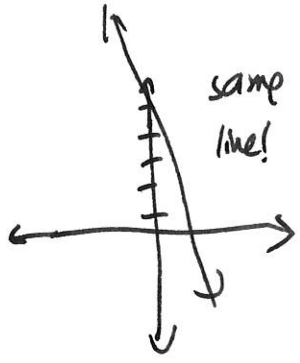
2 eqns in 2 variables (or in fact any number of eqns, any number of vars): there are three ~~solutions~~ possibilities for number of sols:

1) only one solution (our example)

2) infinitely many sols

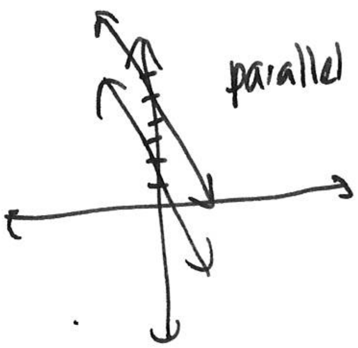
$$2x+y=5$$
$$-4x-2y=-10$$

Same solutions



3) no solutions:

$$\left.\begin{array}{l}2x+y=5\\2x+y=6\end{array}\right\}\text{no way!}$$



Goal:

→ make method work for any # eqns, any number vars.

- figure out how many solutions (0, 1, ∞)

- find all solutions

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algorithm: Gaussian elimination

first thing: some shorthand

$$\left.\begin{matrix} -x+y=3 \\ 2x+y=1 \end{matrix}\right) \rightarrow \left(\begin{array}{cc|c} -1 & 1 & 3 \\ 2 & 1 & 1 \end{array}\right)$$

"augmented matrix"

one row for each eqn

manipulations of eqns ⇝ manipulations of matrix

$$\begin{matrix} -x+y=3 \\ 2x+y=1 \end{matrix} \quad \left(\begin{array}{cc|c} -1 & 1 & 3 \\ 2 & 1 & 1 \end{array}\right) \rightarrow$$

add 2×first row to second row

add 2×first to second

$$\begin{matrix} -x+y=3 \\ 3y=7 \end{matrix} \quad \left(\begin{array}{cc|c} -1 & 1 & 3 \\ 0 & 3 & 7 \end{array}\right) \xrightarrow[\text{by } 3]{\text{divide row 2}} \left[\left(\begin{array}{cc|c} -1 & 1 & 3 \\ 0 & 1 & 7/3 \end{array}\right)\right]$$

this means $y=7/3$

$$\longrightarrow \;.\;.\;.\;.$$

ELEMENTARY ROW OPERATIONS

algebraic manipulation of equations ⟷ changes in rows of corresponding matrix.

three main operations

(I) add a multiple of one equation to another equation ⟷ add a multiple of a row to some other row.

(II) multiply equation by a constant (nonzero) ⟷ multiply all entries of row by a constant

(III) switch two equations in list ⟷ swap two rows

I, II we saw. What's the point of ~~B~~ III?

→ want first row to tell us $x = \dots$, second is $y = \dots$, ...

Goal:

given a system of linear eqns

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right),$$

systematically use row operations to solve

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right) \quad \text{ie} \quad \begin{array}{l} x = a \\ y = b \\ z = c \end{array}$$

8/26

- can't register / stuck in this section?

  email me + UIN: jdl@uic.edu

- HW problems this week posted on site.

a 3×3 system of eqns.

"augmented matrix"

$$\begin{aligned} x+y+z&=6 \\ x-2y+z&=0 \\ z-y&=1. \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

idea: eliminate x, from 2nd and third eqns

eliminate y from 3rd eqn.

then we'll be almost done!

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow[\text{from row 2}]{\substack{\text{subtract} \\ \text{row 1}}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & 0 & -6 \\ 0 & -1 & 1 & 1 \end{array}\right) \xrightarrow[\substack{\text{row 2 by} \\ -1/3}]{\text{multiply}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right)$$

$$\begin{aligned} x+y+z&=6 \\ -3y&=-6 \\ -y+z&=1 \end{aligned} \qquad \begin{aligned} x+y+z&=6 \\ y&=2 \\ -y+z&=1. \end{aligned}$$

i) $x+y+z=6$

ii) $x-2y+z=0$

subtract i: $-3y=-6$.

$$\xrightarrow[\text{to row 3}]{\text{add row 2}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[\text{from row 1}]{\substack{\text{subtract} \\ \text{row 3}}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[\substack{\text{row 2 from} \\ \text{row 1}}]{\text{subtract}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\begin{aligned} x+y+z&=6 \\ y&=2 \\ z&=3 \end{aligned} \qquad \begin{aligned} x&=1 \\ y&=2 \\ z&=3. \end{aligned}$$

this is "triangular"

means x is eliminated from eqn 2 & 3

y is eliminated from 3

solving a general system of linear eqns (§1.2)

two main steps.

1. simplify as much as possible using row operations.
   ("elimating variables"
   "put in echelon form / reduced echelon form")
2. find all sols to simplified equations.

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example:

$$\begin{matrix} x+3z=1 \\ x+y+2z=1 \end{matrix} \rightsquigarrow \begin{matrix} x+3z=1 \\ \underbrace{y-z=0}_{\text{can't simplify further.}} \end{matrix}$$

① done.

z can have any value.

once we choose a value of z, values of x & y are forced on us.

infinitely many solutions!

solution is $\begin{cases} x=1-3z \\ y=z \\ z \text{ is free} \end{cases}$ plug in any z, get a sol!

↖ can be anything

this is ②

Elementary row operations

algebraic manipulations of eqns ⟷ changes in rows of matrix

three legal changes:

(I) adding multiple of one eqn to another eqn ⟷ add multiple of a row of matrix to another row.

(II) multiply equation by a constant ⟷ multiply row by a number

(III) switch ~~to~~ two equations ⟷ swap two rows of matrix.

what's the point of (III)?

want first eqn to tell us x, second to tell us y, ...

e.g.
$$\begin{matrix} y=1 \\ 2x+3y=6 \end{matrix} \rightsquigarrow \begin{matrix} 2x+3y=6 \\ y=1 \end{matrix}$$

goal. given any system of linear eqns, systematically use row operations to get to eqn we can solve.

A matrix is in "echelon form" if

1) any rows of all 0s are at the bottom

2) the leading entry of each row (first nonzero thing) is to right of leading entry of row above

3) all entries in a column <u>below</u> leading entries of a row are 0

*

"reduced echelon form" means

4) leading entry of every row is 1

5) each leading 1 is the only nonzero thing in its column.

<u>Idea:</u> use row operations, get matrix in echelon form. system of equations can get any simpler!

echelon:

$$\left(\begin{array}{ccccc|c} \textcircled{1} & 2 & 2 & 3 & 4 & 1 \\ 0 & 0 & \textcircled{4} & 7 & 1 & 2 \\ 0 & 0 & 0 & \textcircled{5} & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

these leading entries are called "pivot entries".

not reduced!

this one is:

$$\left(\begin{array}{ccccc|c} \textcircled{1} & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & \textcircled{1} & 0 & 3 & 4 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

3) means: if some eqn in list involves x, ~~later~~ subsequent equations don't! so not done simplifying.

how to put a matrix in 'row reduced echelon form

"rref"

1. start with leftmost nonzero column

2. pick a nonzero number in column, & do row exchange to put at top.

$$\left(\begin{matrix}0 & 1 & 2\\ 0 & \cdot & \cdot\cdot\\ \textcircled{1} & 1 & 2\\ 0 & 3 & 4\end{matrix}\right.$$

("move an equation with an x in it to the top")

3. subtract multiples of top row from all other rows to make 0s below top entry.

("eliminate x from all eqns after the first")

4. cover up first eqn in your list, repeat steps 1-3 with remaining equations.

("eliminate y from equations after the second")

(will be in echelon form)

5. start with rightmost pivot, and create 0's above pivot by subtracting ~~mults~~ multiples of this row from rows above.

next time:

1) example of rref

2) finding all solutions from rref.

Announcements

- Piazza up and running!
  (I hope)

- class capacity now 85
  (but full)

- #1.2.4 fixed in typed HW

- Will add some problems from 1.3; won't be on quiz.

- next HW will have a few more problems
  than this one did.

HW due W, quiz on W.

covers previous MWF; will post all by W.

Finding general solution from rref.

a variable not in a pivot column is called a "free variable".

<u>General solution:</u>

- free variables can take any value
  (plug in any number you want)
- other variables determined by the free ones, using eqns from rref.

---

example: rref:

$$\left(\begin{array}{ccc|c} \textcircled{1} & 0 & 3 & 1 \\ 0 & \textcircled{1} & -1 & 2 \end{array}\right) \Rightarrow$$

x, y are pivot variables / basic variable

z is free.

$\uparrow \; \uparrow$ pivots

$\nwarrow$ free variable

$x+3z=1$

$x=1-3z$

$y-z=2$

$$\begin{cases} x=1-3z \\ y=2+z \\ z=\text{free} \end{cases}$$

---

to get a solution, plug in $z=10$.

then $x=1-3(10)=-29$

$y=2+(10)=12$.

$x=-29$, $y=12$, $z=10$ is a solution

3 equations, five variable

$$\left(\begin{array}{ccccc|c} \underline{\underline{1}} & 2 & 1 & 4 & 1 & 4\\ 1 & 2 & 2 & 7 & -2 & -3\\ \underline{\underline{2}} & 4 & -1 & -1 & 0 & 7\end{array}\right) \xrightarrow{\text{add } -1\times\text{row } 1} \left(\begin{array}{ccccc|c} \underline{\underline{1}} & 2 & 1 & 4 & 1 & 4\\ 0 & 0 & 1 & 3 & -3 & -7\\ 2 & 4 & -1 & -1 & 0 & 7\end{array}\right) \xrightarrow[\text{add } -2\times\text{first row}]{}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 4 & 1 & 4\\ 0 & 0 & \underline{1} & 3 & -3 & -7\\ 0 & 0 & \underline{\underline{-3}} & -9 & -2 & -1\end{array}\right) \xrightarrow[\text{add } 3\times\text{row } 1]{} \boxed{\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 4 & 1 & 4\\ 0 & 0 & 1 & 3 & -3 & -7\\ 0 & 0 & 0 & 0 & -11 & -22\end{array}\right)} \xrightarrow[\text{multiply by } -1/11]{\text{echelon form}}$$

leading entry one

$$\left(\begin{array}{ccccc|c} \textcircled{1} & 2 & 1 & 4 & 1 & 4\\ 0 & 0 & \textcircled{1} & 3 & -3 & -7\\ 0 & 0 & 0 & 0 & \textcircled{1} & 2\end{array}\right) \xrightarrow{\text{add } 3\times\text{row } 3} \left(\begin{array}{ccccc|c} 1 & 2 & 1 & 4 & 1 & 4\\ 0 & 0 & 1 & 3 & 0 & -1\\ 0 & 0 & 0 & 0 & \textcircled{1} & 2\end{array}\right) \xrightarrow[-1\times\text{row } 3]{\text{add}}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 1 & 4 & 0 & 2\\ 0 & 0 & \textcircled{1} & 3 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 2\end{array}\right) \xrightarrow{\text{add } -1\times\text{ row } 2} \underbrace{\left(\begin{array}{ccccc|c} \textcircled{1} & 2 & 0 & 1 & 0 & 3\\ 0 & 0 & \textcircled{1} & 3 & 0 & -1\\ 0 & 0 & 0 & 0 & \textcircled{1} & 2\end{array}\right)}_{\text{rref!}}$$

The linear system corresponding to rref is:

- $x_1 + 2x_2 + x_4 = 3$
- $x_3 + 3x_4 = -1$

$x_5 = 2.$

The free variables are? The basic variables are?

Basic: x_1, x_3, x_5 Free: x_2, x_4

The general solution is?

$$\begin{cases} x_1 = 3 - 2x_2 - x_4 \\ x_2 = \text{free} \\ x_3 = -1 - 3x_4 \\ x_4 = \text{free} \\ x_5 = 2 \end{cases}$$

e.g plug in $x_2 = 1$, $x_4 = 2$ — or anything else

$\Downarrow$

$x_1 = -1$

$x_3 = -7$

$x_5 = 2$

plug this into equations: it works!

how many solutions?

0, if rref has a row that looks like

$$(0\ 0\ 0\ 0\ 0 \mid \overset{\swarrow \text{anything not } 0}{3})$$

no solutions to system.

∞ solutions, if there's a free variable.

1 solution: if no free vars, and no row $(0\ 0\ 0\ 0 \mid b)$.

---

the system is "consistent" if there's 1 or ∞ solutions.

not consistent if no sols.

How to check if consistent?

→ figure out rref. is there a row $(0\ 0\ 0\ 0 \mid b)$?

yes: not consistent

no: consistent

vectors

a vector is a matrix with only one column.

$$\vec{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} \quad \vec{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

a vectors with n entries $\Leftrightarrow$ "a vector in $\mathbb{R}^n$"

the "ℝ" is in a funny font: $\mathbb{R}^n$

---

if two vectors are the same size, we can add them.

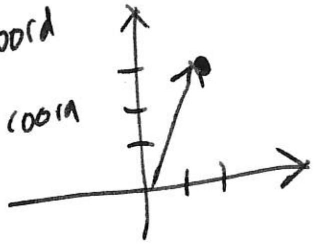
e.g. $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

WARNING
- vector + vector (same size) is OK ✓
- vector + vector (different sizes) nonsense!
- number × vector OK ✓

  eg $5 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$
- number + vector doesn't make sense. (usually)
- vector × vector usually doesn't make sense (except cross product for 3-diml vectors)

think of a vector as a point in 2D or 3D space.

$\begin{pmatrix}2\\3\end{pmatrix}$ ↙ x coord, ← y coord



similarly in 3D.

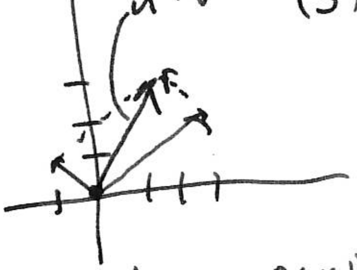
$\begin{pmatrix}1\\2\\3\end{pmatrix}$ ↙ x coord, ← y coord, ← z coord.

$\vec{u}+\vec{v}$, geometrically

$\vec{u}=\begin{pmatrix}3\\2\end{pmatrix}$

$\vec{v}=\begin{pmatrix}-1\\1\end{pmatrix}$

$\vec{u}+\vec{v} = \begin{pmatrix}2\\3\end{pmatrix}$



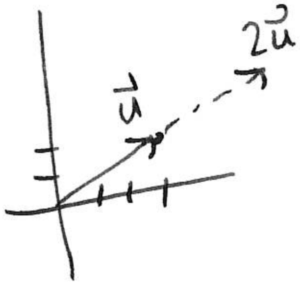
draw a parallelogram

$c\,\vec{u}$, geometrically

↑ number ↖ vector

$\vec{u}=\begin{pmatrix}3\\2\end{pmatrix}$

$2\vec{u}=\begin{pmatrix}6\\4\end{pmatrix}$



Announcements

1) Quiz W on 1.1 & 1.2
(~20 mins)

2) 1.2.2 fixed on HW #1

3) have to change F office hrs time.

Linear combinations

suppose $\vec{v}_1, \dots, \vec{v}_m$ are vectors of the same size.

then

numbers.

$$\vec{y} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m$$

is called a linear combination of $\vec{v}_1, \dots, \vec{v}_m$.

example

$\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

then $5\vec{v}_1 + 2\vec{v}_2 = 5\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ 19 \end{pmatrix}$

is a linear combination of $\vec{v}_1$ & $\vec{v}_2$.

Basic question:

given $\vec{v}_1, \dots, \vec{v}_m$ vectors, and another vector $\vec{y}$,

how to tell if $\vec{y}$ is a linear combo of the $\vec{v}_i$'s?

example

is $\begin{pmatrix}7\\8\\9\end{pmatrix}$ a linear combo of $\begin{pmatrix}1\\2\\3\end{pmatrix}$ & $\begin{pmatrix}4\\5\\6\end{pmatrix}$.

i.e. can I find x_1 and x_2 so that:

$$x_1\begin{pmatrix}1\\2\\3\end{pmatrix}+x_2\begin{pmatrix}4\\5\\6\end{pmatrix}=\begin{pmatrix}7\\8\\9\end{pmatrix}$$

try to solve:

$$\begin{pmatrix}x_1\\2x_1\\3x_1\end{pmatrix}+\begin{pmatrix}4x_2\\5x_2\\6x_2\end{pmatrix}=\begin{pmatrix}x_1+4x_2\\2x_1+5x_2\\3x_1+6x_2\end{pmatrix}=\begin{pmatrix}7\\8\\9\end{pmatrix}.$$

i.e. can we find x_1 and x_2 so

$$x_1+4x_2=7$$
$$2x_1+5x_2=8$$
$$3x_1+6x_2=9$$

---

we know how to tell if linear system has solutions!

$$\left(\begin{array}{cc|c}1&4&7\\2&5&8\\3&6&9\end{array}\right)\xrightarrow[\text{reduction}]{\text{row}}\cdots$$

rref. is there end up with a row

$(0\ 0\,|\,b)$

then no solutions.

moral of this example.

to answer the question; is $\vec{b}$ a linear combo of given vectors $\vec{a}_i$?

$$\Downarrow$$

rephrase this as the "vector equation"

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m = \vec{b}.$$

can we solve for x_i to make this true?

$$\Downarrow$$

has solutions given by augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m & \vec{b} \\ | & | & & | & \end{array}\right)$$

(this matrix has the vectors $\vec{a}_i$ as its columns)

an example

say we have two kinds of chili powder:

SPICY:

$$1T = \underset{\overset{\parallel}{\vec{v}}}{\begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \end{pmatrix}} \begin{matrix} \text{ground pepper} \\ \text{cumin} \\ \text{oregano} \end{matrix}$$

MILD:

$$1T = \underset{\overset{\parallel}{\vec{w}}}{\begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}} \begin{matrix} '' \\ '' \\ '' \end{matrix}$$

what does it mean to take linear combo of $\vec{v}$ & $\vec{w}$?

for example, what's $2\vec{v} + \vec{w}$?

$$2\begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \end{pmatrix} + 1\begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix} \cancel{= \begin{pmatrix} 3/2 \\ 3/2 \\ \end{pmatrix}} = \begin{pmatrix} 5/4 \\ 1 \\ 3/4 \end{pmatrix}$$

2 T spicy + 1 T mild, it contains.

$\underset{5/4}{\cancel{3/2}}$ T ground pepper, $\underset{1}{\cancel{3/2}}$ T cumin, $\underset{3/4}{\cancel{1}}$ T oregano.

A recipe calls for chili powder that's

$$\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \begin{array}{l} \text{pepper} \\ \text{cumin} \\ \text{oregano} \end{array} .$$

Can I make it?

by combining the two I have in some proportion?

$\Downarrow$

tn Can I find x_1 & x_2 so

$$x_1\vec{v} + x_2\vec{w} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} ?$$

equivalently, is $\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$ a linear combination of $\vec{v}$ and $\vec{w}$?

---

we know how to answer this!

Write down augmented matrix

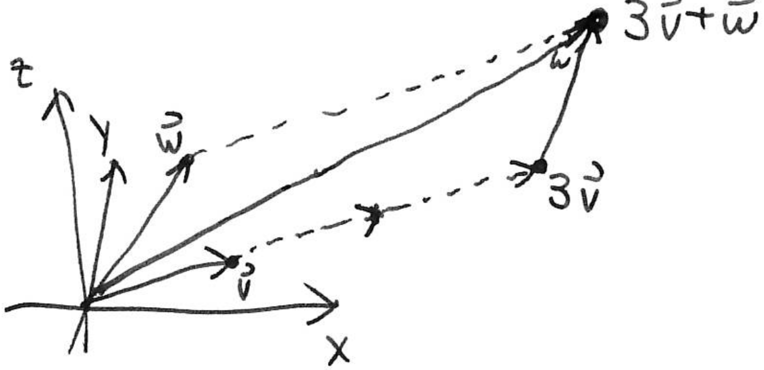
$$\left(\begin{array}{cc|c} 1/2 & 1/4 & 1/3 \\ 1/4 & 1/2 & 1/3 \\ 1/4 & 1/4 & 1/3 \end{array}\right).$$

check if this has solutions.

(it doesn't!)

often it's useful to think of vectors as points in 3D space.

what do linear combinations look like?



what's $3\vec{v}+\vec{w}$?

notice: there's a 2D plane containing $\vec{v}$ & $\vec{w}$.

what happens: any linear combo of $\vec{v}$ and $\vec{w}$ lies in that plane.

(any $a\vec{v}+b\vec{w}$ is in the plane.)

(better picture in book).

More vocab: the <u>span</u> of $\vec{v}_1, \dots, \vec{v}_m$ is the set of all linear combos of the $\vec{v}$'s.

<u>example</u>: the <u>span</u> of two vectors $\vec{v}$ & $\vec{w}$ in 3D is the plane containg those vectors.

suppose A is an m×n matrix, and $\vec{v}$ is a length n vector.

e.g. $\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$ is a 2×3 matrix.

and $\begin{pmatrix}-1\\-3\\7\end{pmatrix}$ is a size 3 vector.

then we can multiply A by $\vec{v}$.

$A\vec{v}$ is another vector, of size m.

How to find $A\vec{v}$: it's a vector obtained by ~~take~~ taking linear combo of columns of A, with coefficients the entries of $\vec{v}$.
/weight

$$\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}-1\\-3\\7\end{pmatrix}=(-1)\cdot\begin{pmatrix}1\\4\end{pmatrix}+(-3)\cdot\begin{pmatrix}2\\5\end{pmatrix}+(7)\cdot\begin{pmatrix}3\\6\end{pmatrix}$$

$$=\begin{pmatrix}-1\\-4\end{pmatrix}+\begin{pmatrix}-6\\-15\end{pmatrix}+\begin{pmatrix}21\\42\end{pmatrix}=\begin{pmatrix}14\\23\end{pmatrix}.$$

Suppose A is given, ($m \times n$ matrix)

and a size m vector $\vec{b}$.

Can you find $\vec{x}$ so that $A\vec{x} = \vec{b}$?

what does this mean?

can we find $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ so that

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b} ?$$

(here $\vec{a}_i$ is the i^{th} column of A)

in other words, is $\vec{b}$ a combo of the colums of A?

to check, write down augmented matrix

$$\left(\underbrace{\begin{array}{cccc} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{array}}_{A} \;\middle|\; \underbrace{\vec{b}}_{\vec{b}.}\right)$$

, run rref.

Quick recap

Three ways to write an system of linear equations.

① $$\left.\begin{aligned} x_1 + 3x_3 &= 7 \\ x_1 + x_2 + x_3 &= 4 \\ 3x_1 - 2x_2 + x_3 &= 0 \end{aligned}\right)$$ "system of linear equations"

② $$x_1\begin{pmatrix}1\\1\\3\end{pmatrix} + x_2\begin{pmatrix}0\\1\\-2\end{pmatrix} + x_3\begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}7\\4\\6\end{pmatrix}$$ "vector equation"

Is $\begin{pmatrix}7\\4\\6\end{pmatrix}$ a linear combination of $\begin{pmatrix}1\\1\\3\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\1\\1\end{pmatrix}$?

③ $$\begin{pmatrix}1&0&3\\1&1&1\\3&-2&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}7\\4\\0\end{pmatrix}$$ "matrix equation"

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$A \qquad \vec{x} = \vec{b}$$

---

same method to solve all 3:

augmented matrix $$\underset{\nwarrow A \quad \uparrow \vec{b}}{\left(\begin{array}{ccc|c}1&0&3&7\\1&1&1&4\\3&-2&1&0\end{array}\right)} \xrightarrow[\text{reduction}]{\text{row}} \left(\begin{array}{ccc|c}1&0&0&1/4\\0&1&0&3/2\\0&0&1&9/4\end{array}\right)$$

1.5: Solution sets of systems of linear equations.

A <u>homogeneous</u> system of linear eqns is one of the form:

$$A\vec{x} = \vec{0} \qquad (\vec{b} \text{ is } 0).$$

$\nwarrow$ the vector $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

<u>example:</u>

$$\underset{\overset{\uparrow}{A}}{\begin{pmatrix}3 & -9 & 6\\ -1 & 3 & -2\end{pmatrix}} \underset{\overset{\uparrow}{\vec{x}}}{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}} = \underset{\overset{\uparrow}{\vec{b}=\vec{0}}}{\begin{pmatrix}0\\0\end{pmatrix}}$$

$$\left.\begin{aligned} 3x_1 - 9x_2 + 6x_3 &= \underline{0} \\ -x_1 + 3x_2 - 2x_3 &= \underline{0} \end{aligned}\right\}$$

for a homogeneous system, these are all 0's

A homogeneous linear system is <u>always</u> consistent.

silly reason: $\vec{x} = \vec{0}$ is a solution.

(but sometimes there are others too!)

$$\left(\begin{array}{ccc|c} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{array}\right) \xrightarrow{\text{row } 1 \div 3} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ -1 & 3 & -2 & 0 \end{array}\right) \xrightarrow[\text{1 to row 2}]{\text{add row}} \left(\begin{array}{ccc|c} \textcircled{1} & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

basic: x_1

free: x_2, x_3. general sol: $\begin{cases} x_1 = 3x_2 - 2x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

---

plug in s for x_2, t for x_3.

our solution is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3s-2t \\ s \\ t \end{pmatrix}$

rewrite this: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

(another way to write down the general solution)

"parametric vector form" for the solution.

what about non-homogeneous systems?

can still write in parametric vector form!

example

$$\left(\begin{array}{ccc|c} 3 & -9 & 6 & 6 \\ -1 & 3 & -2 & -2 \end{array}\right) \xrightarrow{\text{row } 1 \div 3} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ -1 & 3 & -2 & -2 \end{array}\right) \xrightarrow[\text{to row 2}]{\text{add row 1}} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

general solution $\begin{cases} x_1 = 2 + 3x_2 - 2x_3 \\ x_2 = s \\ x_3 = t \end{cases}$.

in parametric vector form,

$$\text{solution is } \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \underbrace{s \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{\text{same as homogeneous sol.}}$$

This always happens!

every solution to $A\vec{x} = \vec{b}$ is of the form

$$\vec{x} = \underset{\substack{\uparrow \\ \text{"particular sol" e.g. } \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}}{\vec{v}_p} + \overset{\swarrow\ s\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.}{\vec{w}} \xleftarrow{} \text{solution to homogeneous } A\vec{x} = \vec{0}.$$

Announcements

- no class M
- quiz W covering MWF homework (including end of 1.3)
  (harder!)
- no office hours today! (sorry)
  Tu 10-11 instead
- quizzes will be returned W.

A couple loose ends.

1. another way to compute $A\vec{x}$, matrix·vector

before: $\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{\vec{x}} = (1)\begin{pmatrix} 1 \\ 4 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 5 \end{pmatrix} + (2)\begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$

new way:

to compute first entry of $A\vec{x}$:

i) take first row of A $\quad (1 \ 2 \ 3)$

ii) pair off entries with $\vec{x}$ $\quad \underline{(1 \ -1 \ 2)}$

iii) multiply corresponding entries; add them up $\quad (1)(1)+(2)(-1)+(3)(2)=5$

(matches earlier answer)

to compute second entry:

same thing, but use second row of A:

row 2 → 4 5 6 → $(4)(1)+(5)(-1)+(6)(2)=11$. ✓

$\vec{x}$ → 1 -1 2

note: the n^{th} entry is the <u>dot product</u> of the n^{th} row of A with the vector $\vec{x}$.

---

this is a little faster to do.

solution set of inhomogeneous linear system.

last time:

$$\left(\begin{array}{ccc|c} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{array}\right) \xrightarrow{\text{rref}\cdots} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{s \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t}_{\text{parameters: what do you plug in for free vars?}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & -9 & 6 & 6 \\ -1 & 3 & -2 & 2 \end{array}\right) \xrightarrow{\text{rref}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Fact: every solution of inhomogeneous system $A\vec{x} = \vec{b}$ can be written as $\vec{x} = \vec{v}_p + \vec{w}$, where $\vec{v}_p$ is some fixed solution of $A\vec{x} = \vec{b}$, and $\vec{w}$ is a solution of $A\vec{x} = \vec{0}$ (involving parameters).

upshot if you know one solution $\vec{v}_p$ (of $A\vec{x} = \vec{b}$), ~~and adding solutions~~ you can find all solutions by adding solutions of $A\vec{x} = \vec{0}$.

in example, we're using $\vec{v}_p = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

what is $\vec{v}_p$ supposed to be?

1) $\vec{v}_p$ could be what we get by solving by row reduction, plugging in 0 for all free variables, and solving basic ones.

(that's what $\begin{pmatrix}2\\0\\0\end{pmatrix}$ is).

2) if you know a solution $\vec{v}_p$ from some other method, you can use it!

---

aren't we getting different answers if we use different $\vec{v}_p$'s?

example: $\left(\begin{array}{ccc|c}3&-9&6&6\\-1&3&-2&2\end{array}\right)$. could use $\vec{v}_p=\begin{pmatrix}2\\0\\0\end{pmatrix}$, or $\vec{v}_p=\begin{pmatrix}5\\1\\0\end{pmatrix}$.

one way: every solution is

$$\begin{pmatrix}2\\0\\0\end{pmatrix}+s\begin{pmatrix}3\\1\\0\end{pmatrix}+t\begin{pmatrix}-2\\0\\1\end{pmatrix} \text{ for some } s,t.$$

other way: every sol is

$$\begin{pmatrix}5\\1\\0\end{pmatrix}+s\begin{pmatrix}3\\1\\0\end{pmatrix}+t\begin{pmatrix}-2\\0\\1\end{pmatrix} \text{ some } s,t.$$

whoh: this looks like two different answers!

It's OK!

these give the same solutions, just for different s&t:

eg. $\begin{pmatrix}4\\2\\2\end{pmatrix}$ comes from first equation, using $s=2$, $t=2$.

$$\begin{pmatrix}2\\0\\0\end{pmatrix}+2\begin{pmatrix}3\\1\\0\end{pmatrix}+2\begin{pmatrix}-2\\0\\1\end{pmatrix}=\begin{pmatrix}4\\2\\2\end{pmatrix}$$

but also comes from second equation, using $s=1$, $t=2$.

$$\begin{pmatrix}5\\1\\0\end{pmatrix}+1\begin{pmatrix}3\\1\\0\end{pmatrix}+2\begin{pmatrix}-2\\0\\1\end{pmatrix}=\begin{pmatrix}4\\2\\2\end{pmatrix}.$$

---

an obvious system:

$x+y=2$

$x+z=2$

~~$y+z=2$~~.

$x=y=z=1$ is a solution

could use that for $\vec{v}_p$.

but row reduction would give something different: $x=2$, $y=z=0$

Applications of linear systems

1. Balancing chemical equations.

$$x_1 CH_3NH_2 + x_2 O_2 \longrightarrow x_3 CO_2 + x_4 H_2O + x_5 N_2$$

each element gives a linear equation!

C: $X_1 = X_3 \rightsquigarrow X_1 - X_3 = 0$

H: $5x_1 = 2x_4$

N: $x_1 = 2x_5$

O: $2x_2 = 2x_3 + x_4$

in matrix form:

$$\begin{array}{c} C \\ H \\ N \\ O \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 5 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 & 0 \end{array}\right)$$

$$\Big\downarrow \text{rref}$$

$$\left(\begin{array}{ccccc|c} \textcircled{1} & 0 & 0 & 0 & -2 & 0 \\ 0 & \textcircled{1} & 0 & 0 & -9/2 & 0 \\ 0 & 0 & \textcircled{1} & 0 & -2 & 0 \\ 0 & 0 & 0 & \textcircled{1} & -5 & 0 \end{array}\right)$$

pivots. x_5 free; call this number s.

$$\begin{cases} x_1 = 2x_5 \\ x_2 = 9/2\, x_5 \\ x_3 = 2x_5 \\ x_4 = 5x_5 \\ x_5 \text{ free} \end{cases}.$$

use $x_5=2$ to make everything integers.

a solution is

$$x_1 = 4$$
$$x_2 = 9 \quad \Rightarrow \boxed{4CH_3NH_2 + 9O_2 \to 4CO_2 + 10H_2O + 2N_2}$$
$$x_3 = 4$$
$$x_4 = 10$$
$$x_5 = 2$$

what if we used $x_5=4$?

$\Downarrow$

$$8CH_3NH_2 + 18O_2 \Rightarrow 8CO_2 + 20H_2O + 4N_2.$$

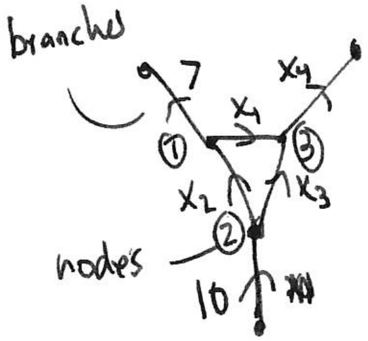
same thing, but doubled!

Applications to networks:

A network is

- a set of "nodes"
- a set of "branches" ~~contain~~ connecting the nodes.

think: network of roads. nodes are intersections, branches of roads connecting the intersections. (one-way roads!)



suppose we know how many cars/min on certain roads. try to figure out how many on other roads

to get linear eqns, two observations:

1) at any intersection, (cars in) = (cars out)

2) (total cars in) = (total cars out)

each intersection gives a linear equation, plus total gives one more

① $x_2 = 7 + x_1$

② $10 = x_2 + x_3$

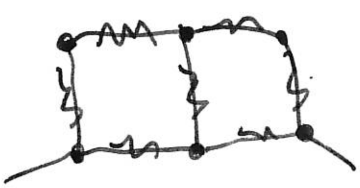
③ $x_1 + x_3 = x_4$

④ $10 = 7 + x_4$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 7 \\ 0 & 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \xrightarrow[\text{rref.}]{\text{solve by}}$$

Network could also be a circuit



label each edge with current

Kirchhoff's junction rule:

(current in) = (current out)

at each nodes

Same analysis!

Announements

New office hours (effective immediately!)

Tu 3-4, online ~7-10 (math310john@gmail.com)

Th 3-4

F 10-11

or email me!

- Pick up your last quiz on your way out

  Grades on Blackboard

- Network problem on HW had a backwards arrow! (Not on quiz)

- Quiz!

Linear independence

A set of vectors $\vec{v}_1, \dots, \vec{v}_n$ is linearly independent

if $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = 0$ has only $x_1 = x_2 = \dots = x_n = 0$.

as a solution.

example (lin. indep.)

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

can we have

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ but x's aren't all 0?

$$= x_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

↑ not equal $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ unless

$x_1 = x_2 = x_3 = 0$.

non-example (not lin. indep)

$$\vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}.$$

$$(1)\vec{v}_1 + (1)\vec{v}_2 + (-1)\vec{v}_3 = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}-1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

---

how to tell if $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are lineerly independent?

ie. is there a solution to the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0} \text{ \underline{other than} } \vec{x} = \vec{0}?$$

to check, write down the corresponding matrix eqn, do row reduction, look for free variable.

Check if

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\3\\5\end{pmatrix}$$ are independent.

$$\Downarrow$$

$$\begin{pmatrix}1&1&1\\1&2&3\\1&3&5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$ have nonzero solution?

$$\left(\begin{array}{ccc|c}1&1&1&0\\1&2&3&0\\1&3&5&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\\1&3&5&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\\0&2&4&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&1&0\\0&1&2&0\\0&0&0&0\end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c}\textcircled{1}&0&-1&0\\0&\textcircled{1}&2&0\\0&0&0&0\end{array}\right)$$

RREF.

general solution:

$$\begin{cases}x_1 = x_3\\ x_2 = -2x_3\\ x_3 \text{ is free}\end{cases}.$$

this has nonzero solutions!

$x_1=1,\ x_2=-2,\ x_3=1$ is a solution

$\Longrightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3$ <u>not</u> linearly independent.

$$(1)\begin{pmatrix}1\\1\\1\end{pmatrix}+(-2)\begin{pmatrix}1\\2\\3\end{pmatrix}+(1)\begin{pmatrix}1\\3\\5\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Say there are only 2 vectors, $\vec{v}_1, \vec{v}_2$. when are they independent?

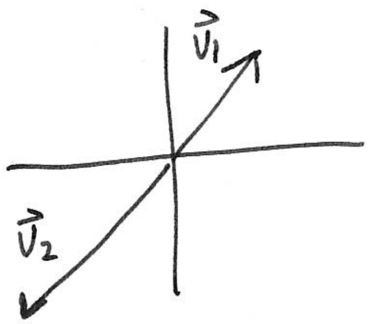
Let's think about when *not* independent:

this means $a_1\vec{v}_1 + a_2\vec{v}_2 = 0$. where a_1, a_2 are not 0.

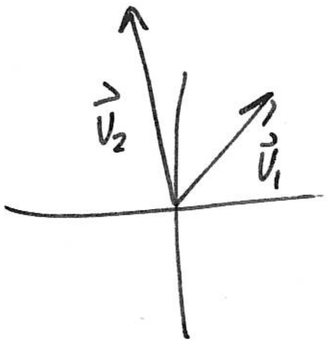
$$\Downarrow$$

$$\vec{v}_1 = -\frac{a_2}{a_1}\vec{v}_2.$$

$\vec{v}_1$ is a multiple of $\vec{v}_2$!



NOT
INDEPENDENT.



INDEPENDENT

what about 3 vectors?

not independent:

$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = 0$, where a's aren't all 0!

$$\Downarrow$$

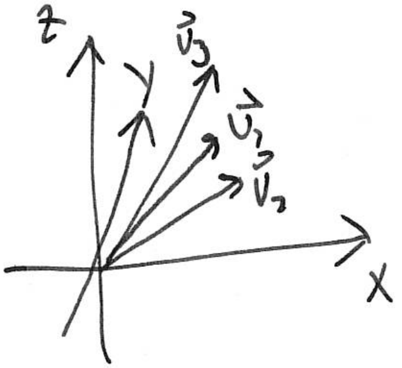
$$\vec{v}_1 = -\frac{a_2}{a_1}\vec{v}_2 \cancel{+} - \frac{a_3}{a_1}\vec{v}_3.$$

$\vec{v}_1$ is a linear combo of $\vec{v}_2$ & $\vec{v}_3$!

(assume 3D vectors!)

Geometrically: linear combos of $\vec{v}_2$ and $\vec{v}_3$ are contained in a plane

so $\vec{v}_1$ is too



LINEARLY DEPENDENT: the three vectors are in a plane

LINEARLY INDEPENDENT three vectors not in a plane.

- Pick up old quizzes after class
- Reminder about parametric vector form, then 1.8
- Attendence sign-in today!

Parametric vector form

example

look at $A\vec{x} = 0$

where $A = \begin{pmatrix} \textcircled{1} & 2 & 0 \\ 0 & 0 & \textcircled{1} \end{pmatrix} \longleftarrow$ augmented matrix for $A\vec{x} = \vec{0}$ is $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

x_1, x_3 basic

x_2 free; call it s

general solution:

$$\begin{cases} x_1 = -2x_2 \\ x_2 = \text{free} \\ x_3 = 0. \end{cases}$$

in vector form:

$\swarrow$ parametric vector form!

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2s \\ s \\ 0 \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

---

example with two free vars:

$$A\vec{x} = \vec{0} \text{ where } A = \begin{pmatrix} \textcircled{1} & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\uparrow$ $\uparrow$
free

~~each~~ plug s for x_2, t for x_3.

parametric vector form! $\downarrow$

$$\begin{cases} x_1 = 2x_2 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$$

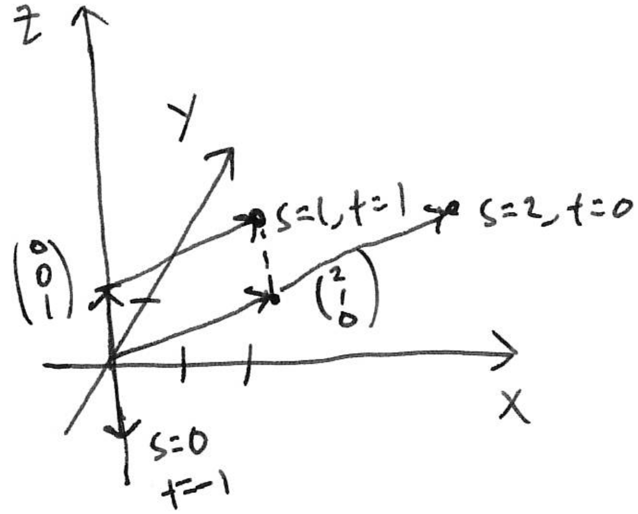
vector form: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2s \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

or: $x_2\begin{pmatrix}2\\1\\0\end{pmatrix}+x_3\begin{pmatrix}0\\0\\1\end{pmatrix}$

---

geometrically:

what do the vectors $s\begin{pmatrix}2\\1\\0\end{pmatrix}+t\begin{pmatrix}0\\0\\1\end{pmatrix}$ look like in 3D?



different s and t give all linear combos of $\begin{pmatrix}2\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$, which is a plane.

the set of all solutions to $A\vec{x}=\vec{0}$, where $A=\begin{pmatrix}1&-2&0\\0&0&0\end{pmatrix}$, is a plane.

(only one free variable → get a line.)

Key facts about linear independence:

1. Definition: $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent if $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = 0$ has only the solution $x_1 = x_2 = x_3 = 0$.

2. How to check: row reduction!

   make a matrix with $\vec{v}_1, \vec{v}_2, \vec{v}_3$ as columns, and a column of 0's.

$$A = \begin{pmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0} \\ | & | & | & | \end{pmatrix}.$$

   (columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3$).

   if any free variables after row reduction, there are solutions.

3. Geometrically: (for three vectors):

   $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly <u>dependent</u> if they are all contained in a plane.

4. if you are looking at vectors in $\mathbb{R}^n$ (n entries), and you have more than n of them, they must be linearly dependent.

Linear transformations

another important source of matrices.

given a $\overset{\text{rows}}{\overset{\downarrow}{m}} \times \overset{\text{column}}{\overset{\swarrow}{n}}$ matrix A, I can as a "function"

$\rightarrow$ A takes in a size-n vector

$\Rightarrow$ A spits out a size-m vector $A\vec{v}$.

example:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{pmatrix}$$

an ~~3×2~~
2×3 matrix.

input: $\underset{\substack{\uparrow \\ \text{size}3}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$, output: $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \underset{\substack{\nwarrow \\ \text{size } 2}}{\begin{pmatrix} 3 \\ 6 \end{pmatrix}}$

input: $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ output: $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Vocab about transformations

a transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$

Is a rule that given a size-n vector, outputs a size-m vector.

example: a matrix A gives a transformation. "matrix transformation"

example: if $m=n=2$, define a transformation that given $\begin{pmatrix} a \\ b \end{pmatrix}$, outputs $\begin{pmatrix} a+1 \\ b+1 \end{pmatrix}$.

$\mathbb{R}^n \leftarrow$ the domain of T

(what do you plug in?)

$\mathbb{R}^m \leftarrow$ the ~~codain~~ codomain of T

(what do get out?)

if $\vec{x}$ is a vector,

$T(\vec{x})$ is called the image of $\vec{x}$.

the set of all possible outputs is called range of T.

not everything in the codomain is a possible output.

<u>example</u> $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathbb{R}^3 \to \mathbb{R}^3$

codomain is $\mathbb{R}^3$.

but: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$.

no matter what I plug in, the output is never $\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$!

$\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ is <u>not</u> in the range.

What is the range of this transformation?

→ all vectors with last entry 0.

(not the same as codomain.)

Questions about a ~~linear~~ matrix transformation:

$$T : \mathbb{R}^n \to \mathbb{R}^m.$$

1) given $\vec{v}$, is $\vec{v}$ in the range of T?
(image)

i.e. does there exist a vector $\vec{x}$ with $A\vec{x} = \vec{v}$

(where A is the matrix defining T.)

we know how to answer that! (row reduction)

e.g.
$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$. is $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in the range?

$\Downarrow$

$A\vec{x} = \vec{v}$

$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right) \xrightarrow{\text{rref}}$ check if the system is consistent.

if consistent, there is a solution, and $\vec{v}$ is in range.

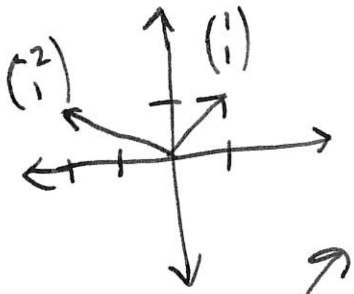
we can draw pictures of ~~linear~~ matrix transformations,

and describe them in words.

e.g. $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$; giving transformation

$T : \mathbb{R}^2 \to \mathbb{R}^2$

draw some vectors



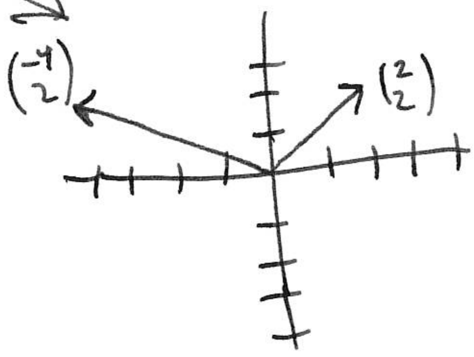
work out

$T(\vec{v}_1)$

draw the outputs

$T\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$



T stretches by a factor of 2.

other matrices also do simple things: eg rotate, reflect.

$$\begin{pmatrix} \textcircled{2} & \textcircled{0} \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \textcircled{-2} \\ \textcircled{1} \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

matrix x vector

- Sub on Wednesday: Chris Skalit from ~~10:10~~
  9:00 AM section.

- quiz on Wednesday: 1.7 & 1.8
  (look at HW!)

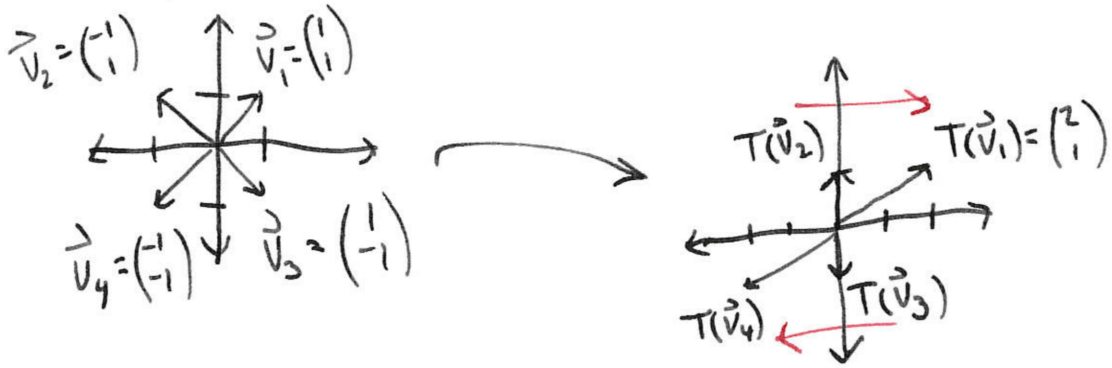
- no office hours tomorrow

Last time: drawing linear transformations.

define $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$

let $T:\mathbb{R}^2\to\mathbb{R}^2$ be the transformation given by $T(\vec{x})=A\vec{x}$.

Let's pick some vectors.



$$T(\vec{v}_1)=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

this is a "shear transformation"

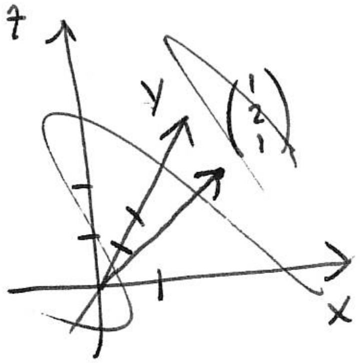
$$T(\vec{v}_2)=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$$

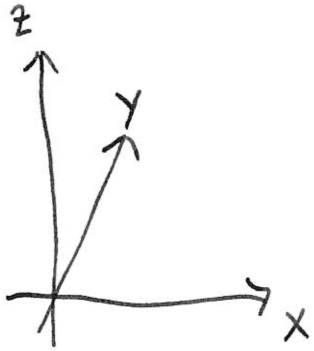
$$T(\vec{v}_3)=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$$

$$T(\vec{v}_4)=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}-2\\-1\end{pmatrix}$$

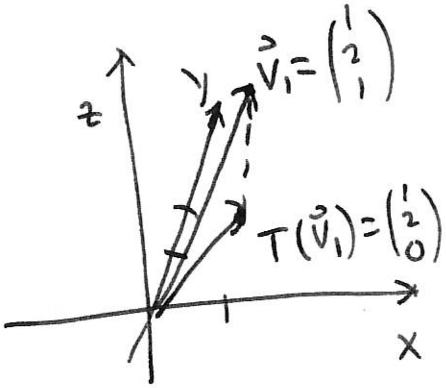
you can try this in 3D too!

$$A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}, \quad T(\vec{v})=A\vec{v}$$





try $\vec{v}=\begin{pmatrix}1\\2\\1\end{pmatrix}$



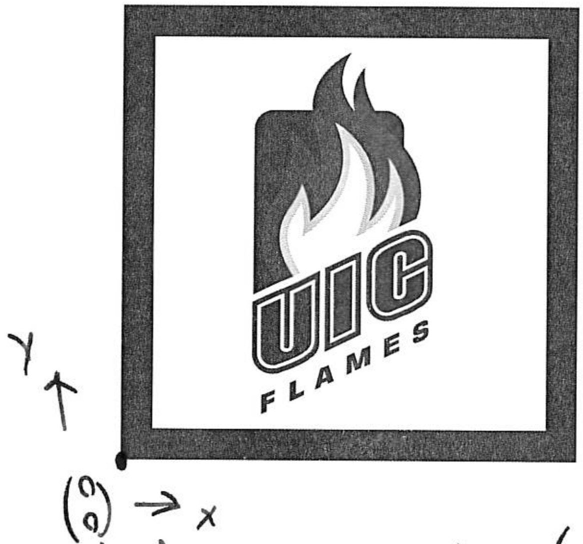
"projection onto
xy-plane"

$$T\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

we can also apply linear transformation to an image: apply T to every pixel.

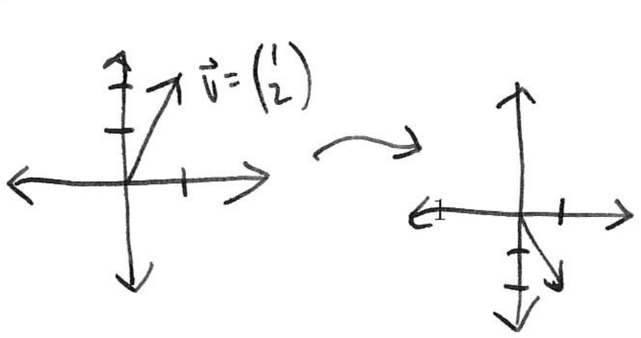
Applying $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$:

ORIGINAL:



What if we apply $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ to this image?

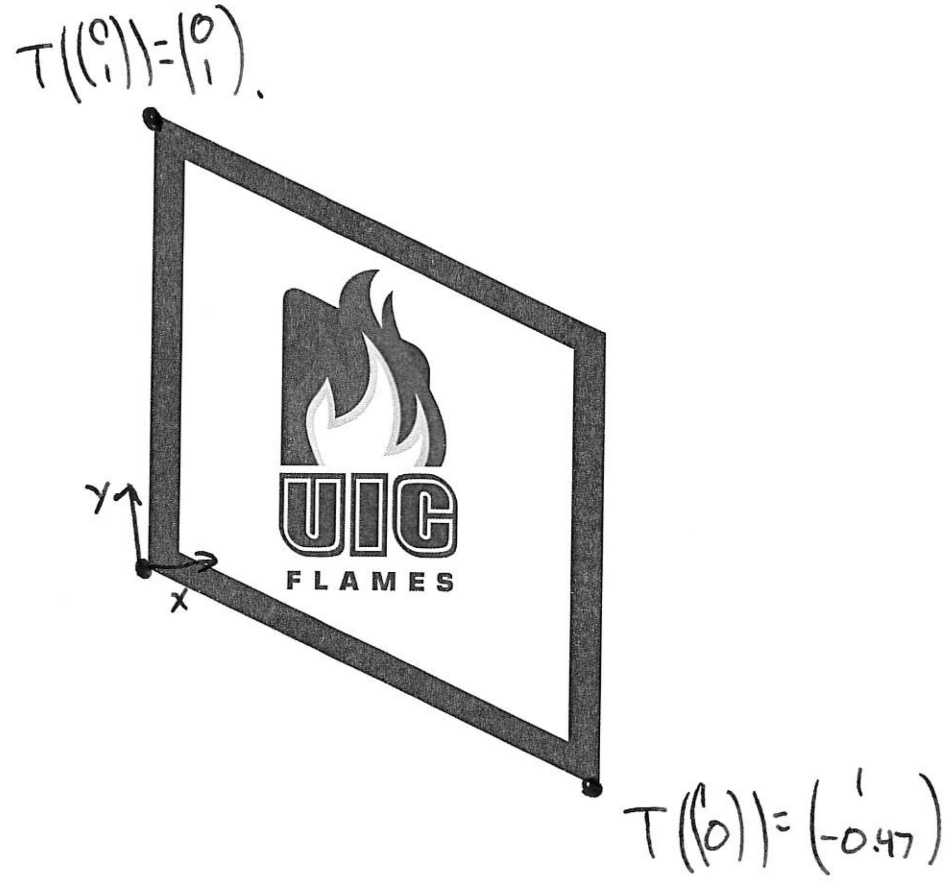
What does A do to a vector?



$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$$

$\Rightarrow$ flip it vertically

Applying $\begin{bmatrix} 1 & 0 \\ -0.47 & 1 \end{bmatrix}$:



Definition:

A linear transformation is a transformation

such that:

1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for any $\vec{u}$ & $\vec{v}$.

"T applied to sum of vectors = T applied to $\vec{u}$, plus T applied to $\vec{v}$."

2) $T(\underset{\substack{\uparrow \\ \text{scalar}}}{c}\overset{\substack{\text{vector} \\ \swarrow}}{\vec{u}}) = c\,T(\vec{u})$.

Example: Any matrix transformation is a linear transformation.

test: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}$ $\quad T : \mathbb{R}^2 \to \mathbb{R}^3$

defined by $T(\vec{x}) = A\vec{x}$

$$\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad \vec{u}+\vec{v} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$T(\vec{u}+\vec{v}) = T\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}.$$

$$T(\vec{u}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$$

$$T(\vec{v}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

$$T(\vec{u}) + T(\vec{v}) = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix} = T(\vec{u} + \vec{v}) \quad ✓$$

A transformation that's <u>not</u> linear:

~~$T(\vec{x})$~~ $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}.$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^2.$

$$T(\vec{u} + \vec{v}) = T\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 16 \end{pmatrix}$$

$$T(\vec{u}) = T\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$$

$$T(\vec{v}) = T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$T(\vec{u}) + T(\vec{v}) = \begin{pmatrix} 1 \\ 9 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}. \neq T(\vec{u} + \vec{v})$$

<u>not linear</u>

Important fact:

if T is a linear transformation, then it's

given by some matrix!

how to find the matrix?

> if $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, it's given by the matrix whose j^{th} column is the vector $T(\vec{e}_j)$, where $\vec{e}_j$ is a vector with j^{th} entry 1 and others 0.

another thing to know:

a transformation is linear if

1) it sends straight lines to straight lines

and

2) it sends $\vec{0}$ to $\vec{0}$.

eg. "rotate 37° counterclockwise"
is transformation $\mathbb{R}^2 \to \mathbb{R}^2$.

clearly linear, but
not clear what
matrix is.

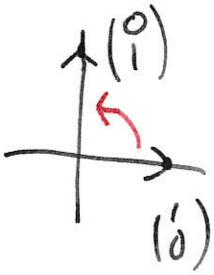
what's the matrix?

let's use "rotate 90° counterclockwise".

want to write down matrix.

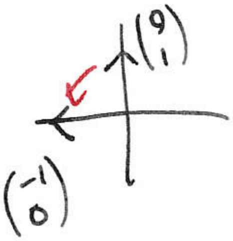
1^{st} column is $T\overset{\swarrow\vec{e}_1}{\begin{pmatrix}1\\0\end{pmatrix}}$

$T\begin{pmatrix}1\\0\end{pmatrix}$ is $\begin{pmatrix}1\\0\end{pmatrix}$ rotated 90°:



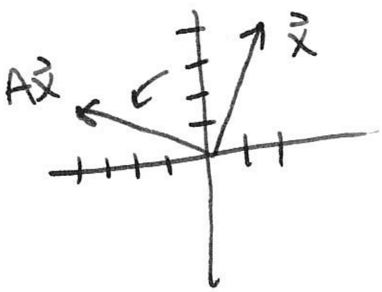
so first column of A is $\begin{pmatrix}0\\1\end{pmatrix}$.

2^{nd} column is $T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$



so $A = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$.

so let's try $\vec{x} = \begin{pmatrix}2\\4\end{pmatrix}$. is $A\vec{x}$ equal to $\vec{x}$ rotated 90°?

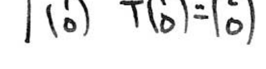


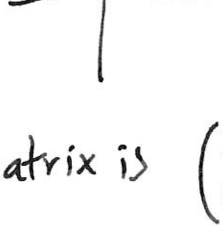
$A\vec{x} = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}-4\\2\end{pmatrix}$

Applying $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$:

want to stretch horizontally by factor of 2.

1^{st} col: $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$ 2^{nd} col: $T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$.





so our matrix is $\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}$.

Applying $\begin{bmatrix} 1 & 0.8 \\ 0 & 1 \end{bmatrix}$:

Applying $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$:

first, some leftovers from §1.9.

VOCAB

Suppose $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

→ T is called onto (surjective) if every $\vec{b} \in \mathbb{R}^m$ is the image of some $\vec{x}$.

→ T is called one-to-one (injective) if every $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x}$.

What these mean and how to check them.

suppose T is given by a matrix A.

Onto. - means that $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$. (maybe infinitely many, maybe just one.)

- which means every $\vec{b}$ is a linear combination of the columns of A, i.e. the columns span $\mathbb{R}^m$.

- to check: run rref on A. the columns of A span $\mathbb{R}^m$ if there is a pivot in every row.

(a fact from §1.4 which John didn't say)

one-to-one -means $A\vec{x}=\vec{b}$ has at most one solution: either 0 or 1, but not infinitely many.

- might as well check for $\vec{b}=\vec{0}$: are there infinitely many solutions to $A\vec{x}=\vec{0}$?

- in other words, are the columns of A linearly independent?

- to check: use row reduction to solve & $A\vec{x}=\vec{0}$. is there a free variable?

  if yes, infinitely many sols, so *not* one-to-one.

Example:

$$A=\begin{pmatrix} \textcircled{1} & 2 & -1 & 4 \\ 0 & \textcircled{1} & 3 & 5 \\ 0 & 0 & 0 & \textcircled{1} \end{pmatrix}.$$

already in echelon form!

onto? yes! pivot in every row.

one-to-one? no! 4 columns, in $\mathbb{R}^3$. can't be independent.

(x_3 is free when we solve $A\vec{x}=\vec{0}$)

§1.10 More applications.

1. Some simple circuits.

a battery and some resistors. let's find the current through each part.

when current passes across a resistor, there's a "voltage drop" $V = RI$. (Ohm's law). some potential is used up.

<u>Kirchhoff's voltage law</u>: if we add up voltage drops around a loop equal to sum of voltage sources in the loop.

in example, top loop

$$\text{voltage drop} \quad 3I_1 + 6I_1 - 6I_2 = 0 \quad \text{no battery}$$

(the middle resistor carries currents from both loops)

bottom loop:

$$1I_2 + 6I_2 - 6I_1 = 2 \quad \text{battery}$$

let's solve.

(

$$\left(\begin{array}{cc|c} 9 & -6 & 0 \\ -6 & 7 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -2 & 0 \\ -6 & 7 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 3 & 2 \end{array}\right)$$

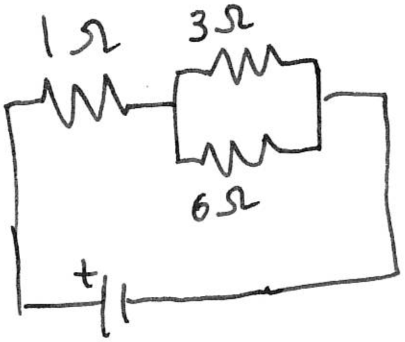
$$\to \left(\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 1 & 2/3 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & 0 & 4/3 \\ 0 & 1 & 2/3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 4/9 \\ 0 & 1 & 2/3 \end{array}\right).$$

$$I_1 = 4/9, \; I_2 = 2/3$$

---

does this agree with physics classes?

same circuit:



resistors in parallel; effective resistance is

$$\frac{1}{\frac{1}{3}+\frac{1}{6}} = \frac{1}{1/2} = 2\,\Omega.$$

in series with $1\,\Omega$:

$$1\,\Omega + 2\,\Omega = 3\,\Omega.$$

current in loop. $V = IR$

$$I = V/R = 2/3 \text{ A.} \;\checkmark$$

Difference equations

- Suppose every year, 5% of Chicagoans move to the suburbs, and 10% of suburbians move to the city.

- Suppose that in year 0 (=2015), there are 3 million people in city limits, 6 million outside.

- how many live in-city in $\overset{\swarrow \text{year } 1}{2016}$? it's given by

$$\begin{pmatrix}0.95 & 0.1\\ 0.05 & 0.9\end{pmatrix}\begin{pmatrix}3\\ 6\end{pmatrix}\begin{matrix}\leftarrow \text{city}\\ \leftarrow \text{suburb}\end{matrix} = \begin{pmatrix}3.45\\ 5.55\end{pmatrix}.$$

in year 2, it's

$$\begin{pmatrix}0.95 & 1\\ 0.05 & 0.9\end{pmatrix}\begin{pmatrix}3.45\\ 5.55\end{pmatrix}$$

in general, $\vec{x}_{k+1} = A\vec{x}_k$. this is an example of a <u>difference equation</u>, and A is the migration matrix. more on this later in the course!

Announcements

- Quiz next W (shorter)

  1.9, 1.10, 2.1

  only circuits

- Midterm a week from that, info coming soon.

Matrix ~~multiplication~~ operations

Some notation.

say A is an $m \times n$ matrix A.

$$\begin{pmatrix} \textcircled{1} & 2 & 3 \\ 4 & \textcircled{5} & 6 \end{pmatrix} \qquad \overset{\text{rows}}{\overset{\downarrow}{2}} \times \overset{\text{columns}}{\overset{\swarrow}{3}}$$

$a_{ij} \leftarrow$ number in i^{th} row, j^{th} column

e.g. $a_{21} = 4$.

$\vec{a}_i = i^{th}$ column vector of A.

e.g. $\vec{a}_3 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

the diagonal entries are a_{jj}

$a_{11} = 1$

$a_{22} = 5$

identity matrix: $\underset{\substack{\uparrow \\ n \times n \text{ matrix; here } n=4.}}{I_n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Addition of matrices

if A & B are two matrices, same size

you can add them: add the corresponding entries

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 1 & 3 \end{pmatrix}$$

if not the same size, can't add!

Scalar multiplication

if A is a matrix, and c is a number, you can multiply: just mutiply every entry.

$$3 \cdot \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 & -3 \\ 6 & 3 & 6 \end{pmatrix}.$$

no big surprises:

$$A+B=B+A$$

$$(A+B)+C=A+(B+C)$$

...

Matrix multiplication (trickier!)

Official definition.

if A $m \times n$ matrix $\Leftarrow$ # ~~no~~ columns of A

B $n \times p$ matrix ||

\# rows of B

we can multiply them (otherwise we can't)

AB is the matrix whose columns are $A\overset{\text{first col of B}}{\vec{b}_1}, A\vec{b}_2, \dots, A\vec{b}_p$.

$$AB = A[\vec{b}_1 \cdots \vec{b}_p] = [A\vec{b}_1 \cdots A\vec{b}_p].$$

example $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & 2 \end{pmatrix}$

\#~~rows~~ cols of A = # ~~cols~~ rows B = 2, so we can multiply.

$$AB = \begin{pmatrix} 1 & -7 & -1 \\ 2 & 11 & 8 \end{pmatrix}$$

$$A\vec{b}_1 = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A\vec{b}_2 = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ 11 \end{pmatrix} \qquad A\vec{b}_3 = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \end{pmatrix}$$

Warnings!

1. $AB \neq BA$ (!)

2. $AB = AC$ does not mean that $B = C$!

   (example on HW)

3. $AB = 0$ can happen even if A and B aren't 0!

$$\underset{\overset{\uparrow}{A}}{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}\underset{\overset{\uparrow}{B}}{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

Some rules of arithmetic are OK:

$A(BC) = (AB)C$ (but not equal $C(AB)$)

$A(B+C) = AB + AC$.

why we can't do BA:

$$\begin{pmatrix}1 & -2 & +1 \\ 0 & 5 & 2\end{pmatrix}\begin{pmatrix}1 & -1 \\ 2 & 3\end{pmatrix}$$

~~first col:~~

$$\xcancel{B\vec{a}_1 \quad \begin{pmatrix}1 & -2 & 1 \\ 0 & 5 & 2\end{pmatrix}\begin{pmatrix}1 \\ 2\end{pmatrix}}$$

can't do this!

The jth column of AB is a linear combo of column~~s~~ of A using weights from jth column of B.

example: 2nd column~~s~~ of AB is linear combo of cols of A, using weights from 2nd col of B $= \begin{pmatrix}-2 \\ 5\end{pmatrix}$:

$$(-2)\begin{pmatrix}1 \\ 2\end{pmatrix} + (5)\begin{pmatrix}-1 \\ 3\end{pmatrix} = \begin{pmatrix}-2 \\ -4\end{pmatrix} + \begin{pmatrix}-5 \\ 15\end{pmatrix} = \begin{pmatrix}-7 \\ 11\end{pmatrix}$$

↑

2nd col of AB.

Fast way to calculate (row-column rule).

The entry in row i, columns j of the product AB can be computed as the sum of the products of corresponding entries in row i of A, col j of B.

$$A=\begin{pmatrix}1 & -1\\ 2 & 3\end{pmatrix} \quad B=\begin{pmatrix}1 & -2 & 1\\ 0 & 5 & 2\end{pmatrix}$$

AB will be a 2x3 matrix!

$$(AB)_{11}=(1)(1)+(-1)(0)=1$$

$$(AB)_{21}=(2)(1)+(3)(0)=2$$

$$\begin{pmatrix}1 & -7 & -1\\ 2 & 11 & 8\end{pmatrix}$$

$$(AB)_{12}=(1)(-2)+(-1)(5)=-7$$

$$(AB)_{22}=(2)(-2)+(3)(5)=11$$

$$(AB)_{13}=(1)(1)+(-1)(2)=-1$$

$$(AB)_{23}=(2)(1)+(3)(2)=8$$

computational complexity for CS types!

~~this~~ $O(n^3)$ to do this method

best is $O(n^{2.3\ldots})$; that's what NumPy etc use.

maybe $O(n^2)$ is possible... we don't know!

transpose of a matrix

"flip it over": if A is $m \times n$ matrix,

then A^T is $n \times m$ matrix,

whose j^{th} row is j^{th} column of A.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

useful facts:

- $(A^T)^T = A$

- $(AB)^T = B^T A^T$

↶↷

switch the order!

matrix powers

if A is a square matrix, we can take powers.

$$A^k = \underbrace{A \cdot A \cdot A \cdots A}_{k \text{ of them}}$$

(if A not square, can't do this)

"fun" example: Fibonacci sequence

$1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 13 \\ 8 \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}.$$

to get the n^{th} Fibonacci number:

compute $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

later in the class:

a formula for n^{th} Fibonacci:

later in class!

$$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

- HW scans posted (sorry!)
- Quiz W: 1.9, 1.10, 2.1 (no migration matrices!)
- Midterm W 9/30: Ch 1, Ch 2.1, 2.2, 2.3
- review sheet & practice test coming W
- Monday: some review - send requests!

2.2 Matrix inversion.

the identity matrix I_n

nxn matrix:

h=3

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Fact 1: $I_n \vec{v} = \vec{v}$ where $\vec{v}$ is any vector

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Fact 2: $I_n A = A$, where A is any matrix with n rows, any number of columns. (call it d)

why?

$$I_n [\vec{a}_1 \cdots \vec{a}_d] = [I_n \vec{a}_1 \cdots I_n \vec{a}_d]$$

$$= [\vec{a}_1 \cdots \vec{a}_d] = A.$$

I_n is the only matrix with these properties!

An $n \times n$ matrix A is invertible if we can find a $n \times n$ matrix C such that $AC = I_n$, $CA = I_n$.

→ A has to be square!

→ not every matrix is invertible!

---

example:

$$A = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}.$$

$$\text{the inverse} = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} = C.$$

$$AC = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}\begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftarrow I_2.$$

$$CA = \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we write $C = A^{-1}$

$$A^{-1} = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}.$$

How to find the inverse:

if A is 2×2:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

if $ad-bc \neq 0$, then A has an inverse.

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

in words: swap top left, bottom right
add – to other two.

(for previous example, $ad-bc=1$.)

(only for 2×2!)

- the number $ad-bc$ is called <u>determinant</u> of A: more later.
- usually $ad-bc \neq 1$, and A^{-1} has fractions
- if $ad-bc=0$, there is no inverse!

<u>ex</u> $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has no inverse!

What's the point.

if A is invertible and $A\vec{x}=\vec{b}$, then:

$$(A^{-1})(A\vec{x})=A^{-1}\vec{b}$$

$$I_n\vec{x}=A^{-1}\vec{b}$$

$$\vec{x}=A^{-1}\vec{b}.$$

> tempting to skip row reduction! but: i) only works when A is invertible, ii) if A bigger then 2x2, finding A^{-1} is more work!

Let's try one: $A=\begin{pmatrix}5&4\\2&2\end{pmatrix}$. solve $A\vec{x}=\begin{pmatrix}1\\2\end{pmatrix}$.

well, $A^{-1}=\frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}=\frac{1}{2}\begin{pmatrix}2&-4\\-2&5\end{pmatrix}=\begin{pmatrix}1&-2\\-1&5/2\end{pmatrix}$

$$\vec{x}=A^{-1}\vec{b}=\begin{pmatrix}1&-2\\-1&5/2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-3\\4\end{pmatrix}$$

Finding inverse of a bigger matrix A an $n\times n$ matrix

- form augmented matrix $[A \mid I_n]$

  I_n: n columns to right of |

- row reduce. if A is invertible, then we get

$$[\underbrace{I_n}_{\text{identity}} \mid \underbrace{B}_{\text{some } n\times n \text{ thing.}}]$$

B is A^{-1} !!

---

example: $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ -3 & -7 & 0 & 1 \end{array}\right] \xrightarrow[\text{to } R_2]{\text{add } 3/2 \cdot R_1} \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 1/2 & 3/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \times 2} \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & 1 & 3 & 2 \end{array}\right] \xrightarrow[\text{to } R_1]{\text{add } -5 \cdot R_2} \left[\begin{array}{cc|cc} 2 & 0 & -14 & -10 \\ 0 & 1 & 3 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \div 2} \left[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \middle| \underbrace{\begin{matrix} -7 & -5 \\ 3 & 2 \end{matrix}}_{B}\right]$$

B is A^{-1}!

There is another way, using "adjoint"/"adjugate" – slower!

Rules for inverses

- $(A^{-1})^{-1} = A.$

  inverse of A^{-1} is $A.$

- $(AB)^{-1} = B^{-1}A^{-1}.$

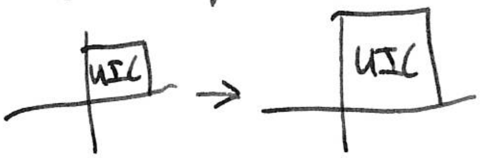
  → if you know inverse A, inverse B, you can find inverse of AB.

- $(A^T)^{-1} = (A^{-1})^T$

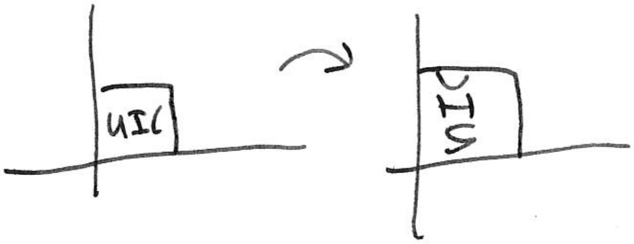
  if you know A^{-1}, can invert $A^T.$

Linear transformations and matrix mult.

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ← expand by factor of 2

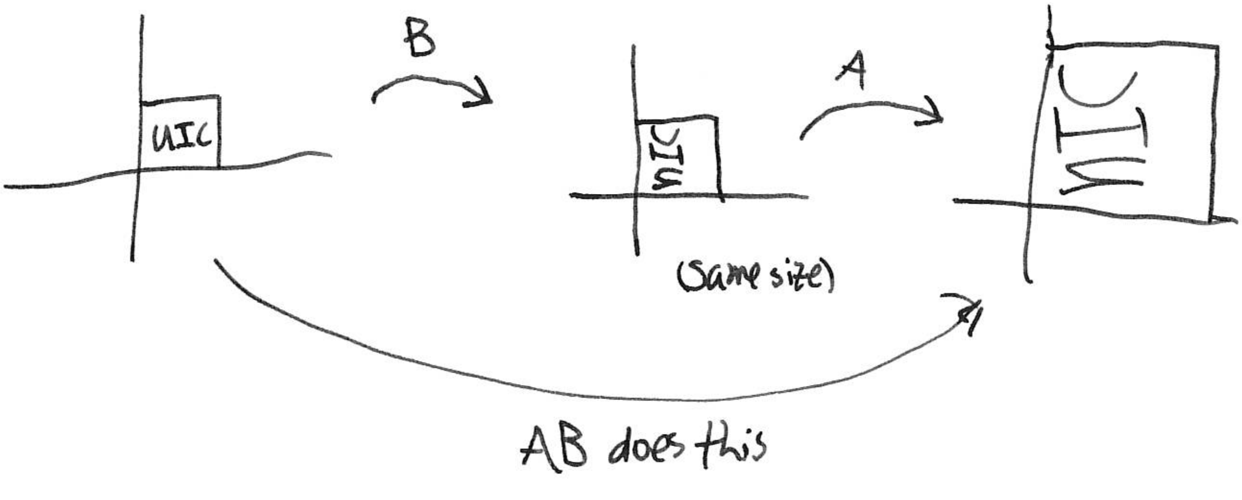


$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ← swap x&y.



in general, AB does

"first do transformation B, then do A".



AB does this

$AB = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$

If A is a matrix giving a transformation, then

A^{-1} gives "undo the transformation".

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ expand by factor of 2

$A^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ shrink by factor of 2.

if $A =$ "rotate 70° counterclockwise"

figure out matrix, find A^{-1}.

$A^{-1} =$ "rotate 70° clockwise".

Exam next W: [illegible]

1.1-1.10, 2.1-2.3 (today)

- review Sheet posted
- practice exams
- HW sols for this week will be posted early

link to old exams: ignore determinants, LU decomposition

will post one with only stuff we did

- M/T go over exam.

How can you tell if a matrix is invertible?

2x2: check if $ad-bc=0$ $\nearrow$ if yes, not invertible

$\searrow$ if no, invertible, and

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Bigger matrices: it's more complicated.

13 ways to tell. (§2.3)

For a given $n \times n$ matrix A, all of the following are all true,

or all are false:

1) A is invertible

2) A is row-equivalent to I_n, the identity

(row reduction on A leads to I_n)

3) A has n pivots

4) $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.

5) Columns are linearly independent

6) the transformation determined by A is one-to-one

7) $A\vec{x} = \vec{b}$ has at least one solution for every $\vec{b}$

8) columns of A span $\mathbb{R}^n$

9) the transformation given by A is onto

10) there is an $n \times n$ C so $CA = I_n$

11) there is an $n \times n$ D so $AD = I_n$.

12) A^T is invertible

* [13) det $A \neq 0$ (we only know this for 2×2)

- understanding why these are the same is good review: how do these fit together?

- some tests will be easier for certain matrices.

* [- if you know A is invertible (check by row reduction), you know all these other things too.

↑ this is the most important: for example, it tells you that if the transformation A is one-to-one, it is onto!

(THIS IS SPECIFIC TO SQUARE MATRICES)

we won't prove it.

Let's just think through an example. and see that everything makes sense.

$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. $ad-bc=0$

most are the same + variations.

1) not invertible!

$ad-bc = (1)(4)-(2)(2) = 0$

[didn't go through this in lecture, but maybe it's useful]

2) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. can't get to I!

remember, if you could,

$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right]$ would you reduce to $[I \mid B]$.

3) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} \textcircled{1} & 2 \\ 0 & 0 \end{pmatrix}$ only one pivot!

4) $A\vec{x} = \vec{0}$ has only $\vec{0}$ sol.

nope! x_2 is free $x_1 = -2x_2$ eg. $(-2, 1)$

$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

5) columns of A linear indep.

nope! we just found a dependence.

$$-2\begin{pmatrix}1\\2\end{pmatrix}+1\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

so not independent.

6) $\vec{x} \mapsto A\vec{x}$ onto one-to-one.

nope! this doesn't work.

$\begin{pmatrix}1&2\\2&4\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix}$ goes to 0: we already found a solution.

7) $A\vec{x}=\vec{b}$ at most one sol for every $\vec{b}$.

nope! two sols for $\vec{b}=0$:

$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}.$$

(in fact, infinitely many: $\vec{x}=s\begin{pmatrix}-2\\1\end{pmatrix}$)

8) columns span $\mathbb{R}^n$.

nope!

$\begin{pmatrix}1&2\\2&4\end{pmatrix} \to \begin{pmatrix}\textcircled{1}&2\\0&0\end{pmatrix}$ no pivot in 2nd row.

the span is just a line.

9) $\vec{x} \mapsto A\vec{x}$ onto? (same thing as columns span)

no: the image is just multiples of $\begin{pmatrix}1\\2\end{pmatrix}$.

10) there's $n \times n$ C so $CA = I$. no!

why?

11) there's $n \times n$ D so $AD = I_n$. no!

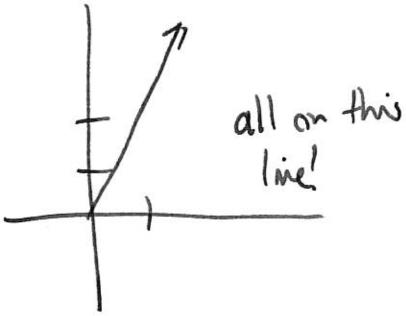
why? $AD = [A\vec{d}_1, A\vec{d}_2]$. each columns is a linear combo of the cols of A.

but $\begin{pmatrix}1\\0\end{pmatrix}$ isn't. so we can't even find D with $AD = \begin{pmatrix}1&?\\0&?\end{pmatrix}$, much less $= Id$.

in terms of linear transformations

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix} \dots$$



no way there can be an inverse transformation, because everything ends up on a single line.

Announcements

- Quiz 3 returned today, Quiz 4 Monday
- HW sols & practice test sols will be posted over the weekend
- Review session (go through practice test):
  - M 9/28, 4-5 SEO 612
  - T 9/29, 5-6? SEO 636
- Tell me ASAP if you need a make-up!
- Attendence sign-in today!

Difference equations

suppose every year

10% of people in City A move to city B

20% of people in City B move to city A

in year 0, both cities have 1,000,000 people.

in year 2, how many will live in each city?

suppose that if year n, the populations are

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{matrix} \leftarrow \text{City A} \\ \leftarrow \text{City B} \end{matrix}$$

in year n+1, it will be

$$\begin{pmatrix} 0.9x_1 + 0.2x_2 \\ 0.1x_1 + 0.8x_2 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{so} \quad \underset{\substack{\uparrow \\ \text{populations in year } n+1}}{\vec{x}_{n+1}} = A\vec{x}_n \swarrow \text{pops in year } n$$

that's what a difference equation is:

$$\vec{x}_{n+1} = A\vec{x}_n \quad \text{(more later)}$$

$$\vec{x}_0 = \begin{pmatrix} 1000000 \\ 1000000 \end{pmatrix}$$

$$\vec{x}_1 = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}\begin{pmatrix} 1000000 \\ 1000000 \end{pmatrix} = \begin{pmatrix} 1100000 \\ 900000 \end{pmatrix}$$

$$\vec{x}_2 = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}\begin{pmatrix} 1100000 \\ 900000 \end{pmatrix} = \begin{pmatrix} 1170000 \\ 830000 \end{pmatrix}$$

year 100?

$$\vec{x}_{100} = \underset{\uparrow}{A^{100}} \vec{x}_0$$

there is a fast way to compute

we'll also see how to compute

$\lim_{n\to 0} \vec{x}_n$ the populations these level off at

"Markov process" use eigenvectors.

Next topic. LU decompostion & Matrix factorizations.

idea: - if you can rewrite a matrix as a product of "simple" matrices, some calculations are simpler.

- many kinds of matrix factorizations, useful for different things: <u>LU decomp</u>, QR decomp, diagonization, singuler value decomp.

scenario: A is a really big matrix, and need to solve

$A\vec{x} = \vec{b}_1, \ A\vec{x} = \vec{b}_2, \ \ldots$ for many different $\vec{b}$'s

lots of wasted effort to do these all separately.

how it works.

given an $m \times n$ matrix A.

(let's say 4×2).

we will find: an $m \times m$ lower triangular matrix L

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{pmatrix}$$

- $m \times n$ matrix U in echelon form

$\uparrow$

not rref!

- $A = LU$.

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix} = \underset{\overset{\uparrow}{L}}{\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}} \underset{\overset{\uparrow}{U}}{\begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \end{pmatrix}}$$

how does this help solve $A\vec{x}=\vec{b}$?

to solve $A\vec{x}=\vec{b}$:

1) solve $L\vec{y}=\vec{b}$ for $\vec{y}$ (easy, since L triangular)

2) then solve $U\vec{x}=\vec{y}$, where $\vec{y}$ is what you found in Step 1. $\vec{x}$ is your answer. (also easy)

why does this work?

$$A\vec{x}=(LU)\vec{x}=L(U\vec{x})=L\vec{y}=\vec{b},$$

like we want.

example

$A = LU$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$

- L is lower triangular ✓
- U is in echelon form ✓
- $LU = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

let's solve $A\vec{x} = \vec{b}$, where $\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

1) first solve $L\vec{y} = \vec{b}$. $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 3 & 1 & 1 \end{array}\right] \xrightarrow[+\text{row}2]{-3\times\text{row}1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \end{array}\right] \text{ done ✓}$$

$$\vec{y} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

2) now solve $U\vec{x} = \vec{y}$ $\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array}\right] \xrightarrow{\text{add row 2 to row 1}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -2 & -2 \end{array}\right] \xrightarrow[\text{by } 2]{\text{divide row 2}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

How to find L and U:

- Matlab, ...

easy version.

- do row reduction on A, using only the operation
  "add something × row i to row j"
- get A to echelon form (not reduced)
  (you might need to swap two rows; in that case,
  LU decomp is more subtle)

- ~~tag~~ LU decomposition is a record of how row reduction
  of A went.

U: the echelon matrix you got at the end
of row reduction.

L: lower triangular matrix with

L_{ij} = what multiple of row j did you
subtract from row i?

diagonal entries 1.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

row reduction:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow[\text{from row 2}]{\text{subtract } 3\times \text{row } 1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 5 & 6 \end{bmatrix} \xrightarrow[\text{from row 3}]{\text{subtract } 5\times \text{row } 1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -4 \end{bmatrix}$$

$$\xrightarrow[\text{from row 1}]{\text{subtract } 2\times \text{row } 2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$ echelon form.

$$u = \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \end{bmatrix}}$$

check:

3×3 $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix}$

$$\overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix}} \overset{u}{\begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}$$

✓

L_{12} = # of row 1's we subtracted from row 2. = 3

L_{13} = # of row 1's subtracted from row 3. = 5

L_{23} = " = 2

Warning: careful with signs. L_{ij} is # row i's <u>subtracted</u>.

multiply $L \cdot U$ is basically running row reduction in reverse: get back A. (details in book)

Warnings.

1) careful with signs
2) some sources all L not to be 1 on diagonal.
3) if need a row swap, this
4) LU decomp is not unique!

Announcements:

- Midterm W 10/10

  (1.1-1.10, 2.1-2.3)

- practice solutions fixed (maybe?)

- review 4-5 today (SEO 612)

  5-6 tomorrow (SEO 636)

Comments on LU.

the way we did it only work if

- all row reduction is of this form:

  subtract c×row i from row j, where row j below row i.

- can't multiply row by a number!

The book's way.

1. Reduce A to echelon form by row operations: this is U.
2. Place entries in L such that the same sequence of row operations reduces L to I.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \xrightarrow[\text{from R2}]{\text{subtract } 3\times R1} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 5 & 6 \end{pmatrix} \xrightarrow[\text{from R3}]{\text{subtract } 5\times R1} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -4 \end{pmatrix} \xrightarrow[\text{from R3}]{\text{subtract } 2\times R2} \overset{U}{\begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}}$$

$$\underset{\text{this is L!}}{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix}} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{''} \underset{I_3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

this lets you multiply rows by numbers to simplify algebra. but we'll end up with non-1's on diagonal.

if A is $m \times n$, U is $m \times n$
L is $m \times m$

---

another warning: - LU not unique: different steps give different valid answers.

- no row swaps, even the book's way: this would make L not triangular

- with row swaps: $PA = LU$
(don't worry too much)

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ (we can't handle this)

§3.1: determinants.

for 2x2 matrices, easy way to check if invertible:

is $ad-bc=0$? [matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$)

if yes, not invertible
if no, invertible.

---

works for bigger matrices! but it's a mess.

given a $\underset{\text{square}}{\underline{\underline{n \times n}}}$ matrix A, the determinant of A

is a number. if 0, then A not invertible
otherwise, A invertible.

| but finding det A is a mess.

two notations in use:

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Where does it come from?

3×3:

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, want to know if invertible.

plan: do row reduction, see if there are three pivots.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \rightarrow \begin{pmatrix} a & b & c \\ ad & ae & af \\ ag & ah & ai \end{pmatrix} \longrightarrow \begin{pmatrix} a & b & c \\ 0 & ae-bd & af-dc \\ 0 & ah-bg & ai-cg \end{pmatrix}$$

$$\longrightarrow \cdots \rightarrow$$

$$\begin{pmatrix} a & b & c \\ 0 & ae-bd & af-dc \\ 0 & 0 & a(\underline{aei+bfg+cdh-bdi-afh-ceg}) \end{pmatrix}$$

if is 0, then A is not invertible

if is not 0, then A is invertible (3 pivots)

that's the determinant!

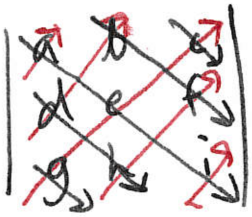
Can do this for $n \times n$ matrices, but it's a sum of $n!$ terms (a mess for $n \geq 4$)

luckily, there are alternative ways to find it

---

how to remember this for 3×3 matrices

det



+ downward diagonals (wrapping around)

− upward diagonals

$$\det A = aei + bfg + cdh - ceg - bdi - afh$$

(same formula from row reduction)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ -1 & 3 & 7 \end{pmatrix}.$$ find $|A|$.

$$= (1)(3)(7) + (2)(5)(-1) + (2)(3)(1)$$

$$-(-1)(3)(1) - (2)(2)(7) - (1)(3)(5)$$

$$= -23$$

$\Rightarrow$ A is invertible!

<u>But</u>: this only works for 3×3!!

Here's a way that works for any n×n matrix

(not the fastest, but sometimes useful)

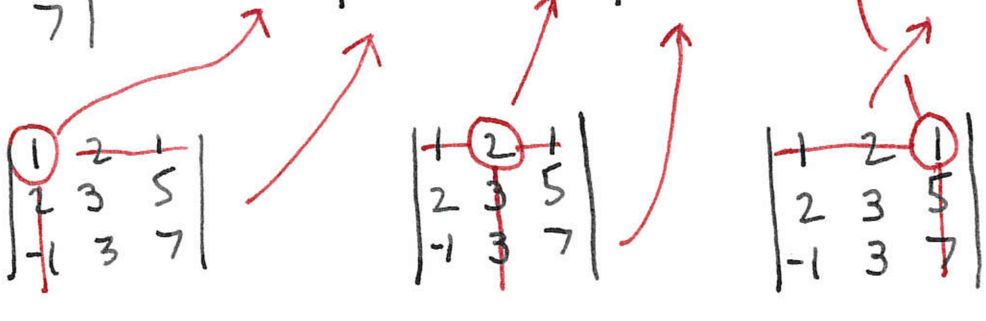
(useful if matrix has many 0's)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ -1 & 3 & 7 \end{pmatrix}.$$

cofactor expansion:

- move across top row of matrix
- for each entry:
  - cross out row & col containing entry, leaving $(n-1)\times(n-1)$ matrix A_{1j}
  - add up entry × $\det(A_{1j})$, alternating signs $+ \; - \; + \; - \; \cdots$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ -1 & 3 & 7 \end{vmatrix} = +(1)\begin{vmatrix} 3 & 5 \\ 3 & 7 \end{vmatrix} - (2)\begin{vmatrix} 2 & 5 \\ -1 & 7 \end{vmatrix} + (1)\begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix}$$



then compute all the smaller determinants!

this takes forever $(O(n!))$, but Friday we'll have another way.

Announcements

- next quiz: 2.5, 3.1, 3.2

  ↑

  LU decomp

- exams returned next week, hopefully

More determinants

so far, only use is to check if A is invertible

painful to compute.

cofactor expansion:

$$\begin{pmatrix} 1 & 0 & 5 \\ 2 & 6 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

you can use any row or columns

not just first row.

but careful with the signs.

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

if doing expansion down second column, alternate + - as usual, but start with -.

$$\det \begin{array}{c} \quad\downarrow\quad \\ \begin{pmatrix} 1 & 0 & 5 \\ 2 & 6 & 2 \\ 0 & 1 & 3 \end{pmatrix} \end{array} = -(0)\begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} + (6)\begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} - (1)\begin{vmatrix} 1 & 5 \\ 2 & 2 \end{vmatrix}$$

$$= 0 + (6)(3) - (1)(-8) = 10.$$

strategy: do expansion on a row or column, where there are lots of 0's

$$\det\begin{pmatrix}1&1&1\\2&2&0\\3&0&0\end{pmatrix} = ?$$

third row: $+(3)\begin{vmatrix}1&1\\2&0\end{vmatrix} - 0\left|\sim\right| + 0\left|\sim\right|$

$= (3)(-2) = -6.$

! In general, to find determinant of a triangular matrix, just multiply the diagonal entries together.

$$\downarrow$$

$$\begin{vmatrix}1&2&3\\0&4&5\\0&0&6\end{vmatrix}$$

expand down first column.

$$+(1)\begin{vmatrix}4&5\\0&6\end{vmatrix} - 0\left|\sim\right| + \left|\sim\right|$$

$$\uparrow$$

still triangular

$= (1)(4)(6) = 24$

Interesting consequence:

If we knew how row operations change det, we could find det of any matrix using row reduction + determinant of echelon form

How do row operations change determinant?

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \det A = ad-bc.$$

Row swap:

$$\begin{pmatrix} c & d \\ a & b \end{pmatrix}, \quad \text{determinant is: } bc-ad$$

$$= -\det(A).$$

→ swapping two rows multiplies det by −1.

Add multiple of row to another row:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow[\text{to row 2}]{\text{add } k\times\text{row1}} \begin{pmatrix} a & b \\ c+ka & \cancel{d+kd} \\ & d+kb \end{pmatrix}.$$

determinant is: $a(d+kb)-b(c+ka)$

~~$=ad+abk-bc-bdk$~~

$=ad+abk-bc-abk$

$=ad-bc$

→ adding multiple of row to another row doesn't change det!

Multiply row by a number:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow[\text{by } k]{\text{mult row 1}} \begin{pmatrix} ka & kb \\ c & d \end{pmatrix}$$

new determinant is $kad - kbc = k(ad - bc)$

→ multiplying a row by k changes determinant by factor of k.

Conclusion: if you can put A into upper triangular form using row swaps, adding mult of row to another (but not multiplying row by number), then

$$\det A = (-1)^{\#\text{row swaps}} \det U$$ ↙ echelon form; triangular

$$= (-1)^{\#\text{row swaps}} (\text{product of pivots})$$

[don't multiply row by number when doing this!
(only need this when trying to get rref

usually this is faster than cofactor expansion

$O(n^3)$ vs $O(n!)$

Example: find det of

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & 3 \\ -2 & 1 & 3 \end{pmatrix} \xrightarrow[\text{to R2}]{\text{add R1}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 5 \\ -2 & 1 & 3 \end{pmatrix} \xrightarrow[\text{to R3}]{\text{add 2R1}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 3 & 7 \end{pmatrix}$$

$$\xrightarrow[\substack{\text{swap} \\ \text{R2, R3}}]{\text{row}} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & 7 \\ 0 & 0 & 5 \end{pmatrix} \text{ done!}$$

det is $(-1)^1(1\cdot3\cdot5)=-15$

we have three methods to find det.

1. 2×2 or 3×3: have a formula
2. cofactor expansion
3. row reduction, multiply pivots.

when to use each?

1. 2×2 or 3×3: use formula
2. bigger, use row reduction
3. matrix with lots of 0's: cofactor expansion

3'. one row with many 0's: cofactor expansion on that row, then other methods for smaller ones.

3''. deriving formulas, proving things: cofactors.

4. Matlab

More facts about determinants.

- $\det(A) = \det(A^T)$

  eg. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, det $= -2$

  $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$, det $= -2$

- if A & B are square, same size

  $\det(AB) = \det(A)\det(B)$.

  $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

  $\det(AB) = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 10 & 14 \end{pmatrix}$

  det: (1) (-2) $(98 - 100 = -2)$

  $\det(BA) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 7 & 18 \end{pmatrix}$

  det $= 3(18) - 7 \cdot 8 = -2$.

  so $\det(AB) = \det(BA)$, even though $AB \neq BA$.

- $\det(A+B) \neq \det(A) + \det(B)$

  doesn't work. please don't do this.

- $\det(-A)$ a little tricky.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$$

$$\begin{vmatrix} -1 & -2 \\ -3 & -4 \end{vmatrix} = -2$$

same

to get $-A$, we multiply n rows by -1.

so det is multiplied by -1 n times:

if n is odd (A is $n \times n$), then $\det(-A) = -\det A$

if n is even $\det(-A) = \det A$.

- what's $\det(A^{-1})$?

we know $A^{-1}A = I_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\det(A^{-1}A) = \det(A^{-1})\det(A) = \det(I_n) = 1$$

$$\det(A^{-1}) = 1/\det(A).$$

- $\det(A^n) = \det(\underbrace{AAAAA \cdots A}_{n \text{ of them}})$

$$= \det(A)\det(A)\det(A)\cdots\det(A)$$

$$= \det(A)^n$$

Announcements:

Quiz W: $\underset{\substack{\uparrow \\ \text{LU!}}}{2.5}$, $\underset{\substack{\curvearrowleft\!\curvearrowright \\ \text{determinants}}}{3.1,\ 3.2}$

we have three ways to compute det.

what's the point?

---

1. cross product.

given two vectors $\vec{v}, \vec{w} \in \mathbb{R}^3$

$\vec{v} \times \vec{w}$ is another vector.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

cofactor in row 1:

$$(v_2 w_3 - v_3 w_2)\hat{\imath} - (v_1 w_3 - v_3 w_1)\hat{\jmath} + (v_1 w_2 - v_2 w_1)\hat{k}$$

this ~~is~~ a little weird - matrix should be a grid of numbers, and that isn't. can't row reduce it.

really just a mnemonic, can expand by cofactors.

---

another interpretation of 3×3 determinants:

$$\det \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{w} \\ | & | & | \end{bmatrix} = \vec{u} \cdot (\vec{v} \times \vec{w}) \qquad \text{(only } 3\times 3\text{!)}$$

"scalar triple product"

why is $\vec{v} \times \vec{w}$ orthogonal to $\vec{v}$?

$$\vec{v} \cdot (\vec{v} \times \vec{w})$$

$$= \det [\underbrace{\vec{v} \quad \vec{v}}_{\text{dependent columns}} \quad \vec{w}]$$

$$\Downarrow$$

$[\vec{v} \ \vec{v} \ \vec{w}]$ not invertible

$$\Downarrow$$

$$\det [\vec{v} \ \vec{v} \ \vec{w}] = 0$$

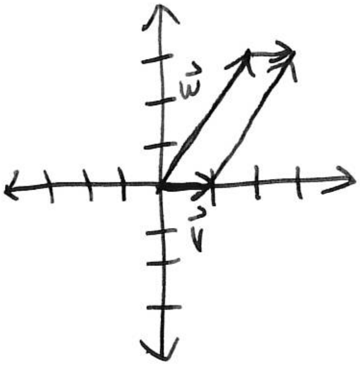
$\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$, so $\vec{v} \perp \vec{v} \times \vec{w}$

Determinants as volume

if A is a 2×2 matrix, det A is the area of parallelogram whose legs are two columns of A.

---

e.g. $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$



$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

det A = 3. area = base × height = 1 × 3 = 3 ✓

---

if A is 3×3, det A is the volume of the parallelpiped spanned by cols of A.

(picture on page 183)

3D version of parallelogram; some kind of "squished cube"

Determinants and linear transformations.

if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, given by a matrix A, and S is a region in $\mathbb{R}^2$ (circle, rectangle, ...).

absolute value of ~~de~~ det

$$\text{area of } \underset{\substack{\uparrow \\ \text{what you get} \\ \text{when you apply} \\ \text{trans to region}}}{T(S)} = \overset{\swarrow}{\left|\det(A)\right|} \cdot \text{area of } S.$$

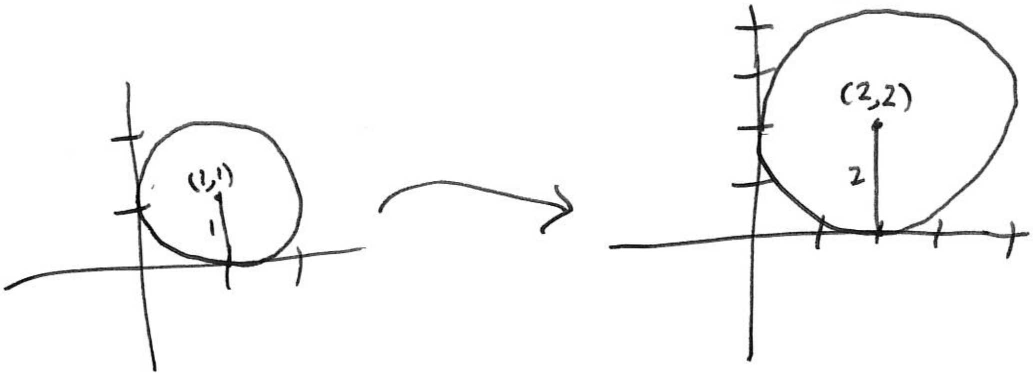
det A = by what factor does T expand area?

example

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \det A = 4.$$

"expand in both directions by factor of 2"

transformation should expand area by factor of 4.



$A = \pi r^2 = \pi$

$A = \pi r^2 = 4\pi$

✓

---

example T = rotate 45° counterclockwise

$$A = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \quad (HW).$$

$$\det A = \frac{1}{2} - -\frac{1}{2} = 1.$$

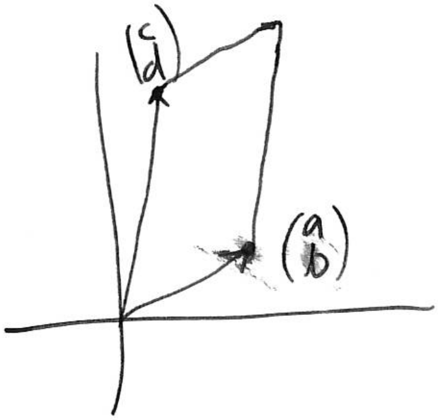
this means area doesn't change, as expected.

✓

Why does this work?

try it when S = unit square =

when we apply T to the square,



$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

we get a parallelogram.

area is $\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}$!

started with area 1,

ended with something of area $\det\begin{pmatrix} a & c \\ b & a \end{pmatrix}$.

---

works the same in $\mathbb{R}^3$: determinant of 3×3 matrix is factor by which transformation increases volumes.

Cramer's rule

- it's another way to solve $A\vec{x}=\vec{b}$ when A is invertible.
- very tempting to use it all the time: it gives a formula for $x_1, x_2, \ldots$
- don't! (very slow)
- but it has its uses: understand how $\vec{x}$ changes when you change A.

---

given an $n \times n$ matrix A, $\vec{b}$ a vector.

write $A_i(\vec{b})$ to be the new matrix where i^{th} col of A is replaced by $\vec{b}$.

then the solution to $A\vec{x}=\vec{b}$ is given by

$$x_i = \frac{\det A_i(\vec{b})}{\det A}$$

consider the system

$$\begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 - \underset{\nwarrow \text{parameter}}{s} x_2 = 3 \end{cases} \qquad \underset{\underset{A}{\uparrow}}{\begin{pmatrix} 1 & 3 \\ 2 & -s \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underset{\underset{\vec{b}}{\uparrow}}{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

how does the solution depend on s? for what s is there a solution?

|| could just compute A^{-1} in terms of s, and solve in terms of s.

$$x_1 = \frac{\det A_1(\vec{b})}{\det A} = \frac{\det \begin{pmatrix} 1 & 3 \\ 3 & -s \end{pmatrix}}{\det A} = \frac{-s-9}{-s-6}$$

$$x_2 = \frac{\det A_2(\vec{b})}{\det A} = \frac{\det \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}}{\det A} = \frac{1}{-s-6}.$$

Another way to find A^{-1}

(Slow for big matrices, OK for 3x3)

the (i,j)-entry of A^{-1} is given by

$$\frac{\det(A_i(\vec{e}_j))}{\det(A)} .$$

(you could find A^{-1} by doing this for every entry)

(Slow; but quick to find specific entry)

(Just a case of Cramer's rule).

$A=\begin{pmatrix}2&7&3\\2&1&2\\3&4&5\end{pmatrix}$. what's the top left entry of A^{-1}?

use cofactors on col 1 ↓

it's $$\frac{\det(A_1(\vec{e}_1))}{\det A} = \frac{\det\begin{pmatrix}1&7&3\\0&1&2\\0&4&5\end{pmatrix}}{\det\begin{pmatrix}2&7&3\\2&1&2\\3&4&5\end{pmatrix}} = \frac{1\cdot\begin{vmatrix}1&2\\4&5\end{vmatrix}}{-19} = 3/19$$

- Exam 1 will be returned Friday - sorry!
- Today: one more thing in 3.3, then start Ch. 4.

Inverses using Cramer's rule.

last time:

replace i^{th} col of A with $\vec{e}_j$.

$$(i,j)\text{-entry of } A^{-1} = \frac{\det(A_i(\vec{e}_j))}{\det A}$$

we can simplify this a little more:

i^{th} column has only one nonzero entry. Cofactor expansion down i^{th} column gives:

$$\det(A_i(\vec{e}_j)) = (-1)^{i+j} \det A_{ji}$$

cross out j^{th} row, i^{th} column

$$= C_{ji}$$, called a "cofactor".

so:

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

each C_{ij} is a det!

$$A = \begin{pmatrix} 1 & 1 & 7 \\ 1 & 2 & 8 \\ 0 & 3 & 9 \end{pmatrix}.$$

what's the (2,2)-entry of A^{-1}?

$$C_{22} = (-1)^4 \det\begin{pmatrix} 1 & 7 \\ 0 & 9 \end{pmatrix} = 9.$$

so (2,2)-entry is

$$\frac{C_{22}}{\det A} = \frac{9}{6} = \boxed{\frac{3}{2}}$$

§4.1.

A vector space is any collection of objects you can add, and multiply by scalars, in a way that obeys the "usual rules of arithmetic":

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(10 rules needed)

Examples:

1. vectors in $\mathbb{R}^n$
2. all quadratic polynomials in a variable x. (ax^2+bx+c).

   if we add two of those, we get another:

   $(x^2+2x+3) + (-7x^2 + 3x - 4) = (-6x^2+5x-1)$
3. 2x3 matrices
4. all functions of one variable: $\sin(x)$, e^x, $2+x^2$, ...

What's the point?

a lot of the stuff we've been with vectors, works just as well with other vector spaces.

example: $\overset{\text{taking derivative}}{\swarrow}$

$$\frac{d}{dx} : \text{Functions in one var} \to \text{Functions in one var}$$

is a linear transformation!

$$\frac{d}{dx}(f+g) = \frac{d}{dx}f + \frac{d}{dx}g.$$

$$\frac{d}{dx}(cf) = c \cdot \frac{d}{dx}f.$$

Q. is $\frac{d}{dx}$ a one-to-one transformation?

NO:

$\underline{\underline{\frac{d}{dx}f = x^2}}$ has many solutions:

$$\frac{x^3}{3},\ \frac{x^3}{3}+7,\ \frac{x^3}{3}+c.$$

$$\underset{\substack{\uparrow \\ \text{derivative}}}{A}\ \underset{\substack{\uparrow \\ \text{function}}}{\vec{x}} = \underset{\substack{\uparrow \\ \text{another function.}}}{\vec{b}}$$

for $\vec{b} = x^2$ there are many $\vec{x}$ that work.

Subspace

a subspace of a vector space is a collection of

of objects such that - when you add 2, you get another

- when mult. by scalar, you get another.

1) vector space $= \mathbb{R}^3$

consider all vectors with $X_1 = X_2 = X_3$. is it a subspace?

e.g. $\begin{pmatrix}2\\2\\2\end{pmatrix} + \begin{pmatrix}3\\3\\3\end{pmatrix} = \begin{pmatrix}5\\5\\5\end{pmatrix}$

$-1 \begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}-2\\-2\\-2\end{pmatrix}$ ✓

2) vector space $= \mathbb{R}^3$

consider all vectors with $x_1 = 1$. is this a subspace?

no: $\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\end{pmatrix}}_{\text{these are both in my collections}} = \underbrace{\begin{pmatrix}2\\0\\0\end{pmatrix}}_{\text{but this isn't.}}$

Announcements

- Exams back today

- last quiz

- today: more of §4.

Vector space:

a set of things you can add and multiply by scalers in a way obeying usual rules of algebra.

(regular vectors in $\mathbb{R}^n$, m×n matrices, quadratic polynomials, ...)

A **subset** is a bunch of items inside a vector space, defined by having some property

(all vectors of length < 1, all vectors with first entry 0, all matrices with det = 0, ...)

A **subspace** of a vector space is a subset for which:

- sum of two things in subset is still in subset
- scalar × thing in subset is still in subset.

↖ (allowed to be 0 or negative)

Examples

Vector space $= \mathbb{R}^3$ (regular vectors)

1. vectors with last entry 0: $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix} \checkmark$$

both in subset

$$(-7)\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -14 \\ 0 \end{pmatrix}. \checkmark$$

yes, subspace

2. vectors with length less than 1.

not a subspace:

$$\underset{\text{in subset}}{\begin{pmatrix} 3/4 \\ 0 \\ 0 \end{pmatrix}} + \underset{\text{in subset}}{\begin{pmatrix} 3/4 \\ 0 \\ 0 \end{pmatrix}} = \underset{\substack{\text{length} = 3/2 \\ \underline{\text{not}} \text{ in subset}}}{\begin{pmatrix} 3/2 \\ 0 \\ 0 \end{pmatrix}}$$

so <u>not</u> a subspace.

If you can find example $\vec{v}, \vec{w}$ with $\vec{v}$ and $\vec{w}$ in subset, but $\vec{v}+\vec{w}$ not, you know it's not a subspace.

3. vectors whose entries are all positive.

subspace?

no: $$\underset{\substack{\uparrow\\ \text{scalar}}}{(-1)} \cdot \underset{\substack{\uparrow\\ \text{in subset}}}{\begin{pmatrix}2\\3\end{pmatrix}} = \underset{\substack{\uparrow\\ \text{not in subset.}}}{\begin{pmatrix}-2\\-3\end{pmatrix}}$$

2x2 matrices as vector space

1. matrices with left column $= 0$.

$$\begin{pmatrix}0 & a\\0 & b\end{pmatrix}$$

subspace ✓

2. matrices with determinant 0.

$$\underset{\substack{\det = 0\\ \text{in subset}}}{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}} + \underset{\substack{\det = 0\\ \text{in subset}}}{\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}} = \underset{\substack{\det = 1\\ \text{not in subset}}}{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}$$

Quadratic polynomials as vector space

$f(t) = at^2 + bt + c$

1. polynomials with constant term 0 $(c=0)$

   subspace ✓

2. polynomials with $f(1)=0$.

   e.g. t^2-3t+2 ✓

   add two together, get another; same for × scaler.

   subspace ✓

3. polynomials with $f(1)=1$.

   <u>no</u> ~~(t^2+1)~~

$$\underset{\substack{\uparrow \\ \text{plug in} \\ \text{1, get 1}}}{(t^2)} + \underset{\substack{\uparrow \\ \text{plug in } t=1, \\ \text{get 1}}}{(t^2-3t+3)} = \underset{\substack{\uparrow \\ \text{plug in } t=1, \\ \text{get 2.}}}{2t^2-3t+3}$$

Back to $\mathbb{R}^3$.

a subspace:

the span of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$

i.e. all possible linear combinations of these two vectors. $s\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}1\\2\\3\end{pmatrix}$.

why is sum of two vectors in span still in span?

$$\left[3\begin{pmatrix}1\\1\\1\end{pmatrix}+5\begin{pmatrix}1\\2\\3\end{pmatrix}\right]+\left[(-2)\begin{pmatrix}1\\1\\1\end{pmatrix}+2\begin{pmatrix}1\\2\\3\end{pmatrix}\right]$$

$=\begin{pmatrix}8\\13\\18\end{pmatrix}$, in span. $\qquad =\begin{pmatrix}0\\2\\4\end{pmatrix}$, in span

sum is

$$(3-2)\begin{pmatrix}1\\1\\1\end{pmatrix}+(5+2)\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}8\\15\\22\end{pmatrix}\text{ in span.}$$

this <u>is</u> a subspace.

Nothing special about these vectors!

in $\mathbb{R}^n$, given $\vec{v}_1, \ldots, \vec{v}_r$ vectors, the span is always a ~~sp~~ subspace of $\mathbb{R}^n$.

More subspaces of $\mathbb{R}^n$: subspaces related to a matrix.

given an $m \times n$ matrix A, we are going to define

- Nul A ("nullspace" of A): subspace of $\mathbb{R}^n$.
- Col A ("column space" of A): subspace of $\mathbb{R}^m$.

the nullspace of A is the set of all vectors with $A\vec{x} = 0$

(all solutions to homogeneous linear system $A\vec{x} = \vec{0}$)

if A is eg. 2×3 $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, this is bunch of size-3 vectors.

$\Downarrow$

subspace of $\mathbb{R}^3$

---

the column space of A is the span of the columns of A.

Given A, how do you check if $\vec{x}$ in nullspace?

compute $A\vec{x}$. if $\vec{0}$ yes. if not, not in null space.

How to find entire nullspace?

solve $A\vec{x}=\vec{0}$, put in parametric vector form.

nullspace is the span of all vectors appearing in PVF.

$$A=\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ -1 & -2 & 2 & 3 \end{pmatrix}$$

general sol is $A\vec{x}=\vec{0}$ $\quad s\begin{pmatrix}-2\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\-2\\1\end{pmatrix}$

~~null~~ nul $A=\text{span}\left(\begin{pmatrix}-2\\1\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\-2\\1\end{pmatrix}\right)$.

Given A, how to check if $\vec{b}$ is in column space?

$\Updownarrow$

does $A\vec{x}=\vec{b}$ have any solutions?

if yes, $\vec{b}$ in Col A

if no, $\vec{b}$ isn't

---

the set of all vectors of the form $\begin{bmatrix} 6s+4t \\ 2s+5t \\ 3s+6t \end{bmatrix}$

is the column space of <u>some</u> matrix. Which one?

this is the same as $s\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

so all linear combos of $\begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$.

$$A=\begin{bmatrix} 6 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

The nullspace makes sense for other vector spaces too.

e.g. D : function of one var → functions of one var

$$D(f) = f'' + \omega^2 f.$$

this is a linear transformation

the nullspace of D is all f with $D(f) = 0$

all solutions to the differential equation

$$f'' + w^2 f = 0.$$

- Quiz W (3.3, 4.1, 4.2)

- some notes:

① Transpose when using Cramer's rule.

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{31} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

C_{ij} are "cofactors": $C_{ij} = (-1)^{i+j} \cdot \det\begin{pmatrix}\text{matrix you get} \\ \text{when delete row } i, \\ \text{col } j\end{pmatrix}$

if you want row 2, col 1 of A^{-1}, cross out row 1 col 2

and add a sign.

you can always check if your inverse is right: compute AA^{-1}, should be I_n.

see page 181 for an example.

(easy to end up with $(A^{-1})^T$ instead.)

② LU decomposition for $m \times n$ matrices.

L is square, $m \times m$

U is $m \times n$

e.g. $$\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix} = \underset{\underset{L}{\uparrow}}{\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}} \underset{\underset{U}{\uparrow}}{\begin{pmatrix} * & * \\ * & * \\ * & * \end{pmatrix}}$$

should get $A=LU$

③ last week I said quadratic polynomials ke are a vector space:

$$at^2+bt+c.$$

This must include~~s~~ polynomials where $a=0$! $t+1$ is in space

really want: polynomials of degree ≤ 2, not necessarily equal to 2.

eg. $(t^2+1)+(-t^2+6+3t)=3t+7$.

④ Two (slightly} weird examples of subspaces)

in $\mathbb{R}^n$ (or any other vector space).

I. the zero subspace: the only vector in it is $\vec{0}$.

- $\vec{0} + \vec{0} = \vec{0}$
- $c\vec{0} = \vec{0}$.

important to include; not just semantics:

$Nul(I_3)$ = set of all vectors with $I_3\vec{x} = \vec{0}$.

the only such $\vec{x}$ is $\vec{0}$!

so $Nul(I_3)$ = subspace containing only $\vec{0}$.

II. $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

$Col(I_3)$ is all of $\mathbb{R}^3$.

4.3 Bases

(this is plural of basis)

A collection of vectors in a vector space or subspace is a basis if the following are true:

1) the vectors are linearly independent:

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$$

has only the solution $c_1 = \cdots = c_p = 0$.

2) the $\vec{v}_i$'s span the space/subspace.

these are in tension: linearly independent means not too many vectors, but spanning means not too few.

Examples.

1) $\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix}$ is a basis for $\mathbb{R}^3$.

- linearly independent ✓
- they span $\mathbb{R}^3$: any vector in $\mathbb{R}^3$ is a combination of these:

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

if we added a 4th vector, it wouldn't work: they'd be dependent.

if we get rid of any of the three, they don't span!

$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$: $\begin{pmatrix}1\\2\\3\end{pmatrix}$ isn't a linear combo of these.

2. Space of 2×2 matrices.

a basis is some set of matrices that area independent, and any 2×2 matrix is a combination.

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis.

- they span:

$$\underset{\substack{\uparrow \\ \text{random} \\ 2\times2 \text{ matrix}}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- linearly independent.

3. quadratic polynomials.

$t^2, t, 1$ is a basis.

eg. $3t^2 - 7t + 6 = (3)\cdot t^2 + (-7)\cdot t + (6)\cdot 1$

Bases for our favorite subspaces: Nul A, Col A

where A is a matrix.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & 1 \\ -2 & -4 & 3 \end{pmatrix}.$$

Let's find a basis for Nul A. solve $A\vec{x} = \vec{0}$.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -2 & 1 & 0 \\ -2 & -4 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ -2 & -4 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

general sol is:

$$\begin{cases} x_1 = -2x_2 \\ x_2 \text{ is free} \\ x_3 = 0 \end{cases} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2s \\ s \\ 0 \end{pmatrix} = s \underline{\underline{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}}$$

> basis for Nul A is all vectors that show up in parametric vector form.

in this case, only one vector: everything in Nul A is a multiple of this.

in general, one vector in the basis per free variable.

Finding a basis for Col A.

basis is not just the columns of A: they won't be linearly indep.

solution: get rid of some columns; then they'll be independent, and still span. which ones to keep to get a basis?

$\Rightarrow$ use the pivot columns.

these give a basis for Col A.

Catch: need columns from A itself, not rref(A):

do row reduction, get to echelon form. find which # columns are pivots, then pick those cols from A. the are a basis for Col A.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & 1 \\ -2 & -4 & 3 \end{pmatrix} \quad \text{rref is} \quad \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

pivot cols are column 1, column 3.

a basis for Col A is $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Given a bunch of vectors, (not independent),

how to find a basis for ~~t~~ span$(\vec{v}_1, \dots, \vec{v}_d)$?

solution: put the vectors as columns of matrix,

and find basis for Col A. (this will be some subset

of your vectors; get rid of redundant ones)

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \begin{pmatrix}1\\2\\3\end{pmatrix} \begin{pmatrix}2\\3\\4\end{pmatrix} \begin{pmatrix}3\\5\\7\end{pmatrix}$$

stick in a matrix, row reduce: answer: the pivot cols.

Bases as coordinate systems

Fact: if $\vec{v}_1, \dots, \vec{v}_d$ is a basis for a vector space / subspace, then any vector $\vec{x}$ can be written as a combination of $\vec{v}_1, \dots, \vec{v}_d$ in exactly one way:

$$\vec{x} = c_1 \vec{v}_1 + \dots + c_d \vec{v}_d$$

Why? $\vec{v}_1, \dots, \vec{v}_d$ are a basis for the vector space V. that means that they span V, which means

$\vec{x} = c_1 \vec{v}_1 + \dots + c_d \vec{v}_d$ for some values of c_i.

why is this unique? well, suppose

$\vec{x} = c_1 \vec{v}_1 + \dots + c_d \vec{v}_d$

and $\vec{x} = e_1 \vec{v}_1 + \dots + e_d \vec{v}_d$.

that means $\vec{0} = (c_1 - e_1)\vec{v}_1 + \dots + (c_d - e_d)\vec{v}_d$.

$\vec{v}_i$'s are a basis so linearly independent!

so it must be that $c_1 - e_1 = 0$, $c_2 - e_2 = 0, \dots$ $c_d - e_d = 0$

ie $c_1 = e_1, \dots, c_d = e_d$

If $\mathcal{B} = \vec{b}_1, \ldots, \vec{b}_n$ is a basis for a vector space V

and $\vec{v}$ is a vector, then

$$\vec{v} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n \text{ in exactly one way.}$$

the weights $c_1, \ldots, c_n$ are called the

$\mathcal{B}$-coordinates of $\vec{v}$.

write $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. this is the coordinate vector for $\vec{v}$.

Example: vector space $\mathbb{R}^2$.

basis $\mathcal{B}$: $\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$

$$\begin{matrix}\uparrow & \uparrow\\ \vec{b}_1, & \vec{b}_2.\end{matrix}$$

this is a basis ✓.

the vector $\vec{v}=\begin{pmatrix}11\\7\end{pmatrix}$ can be written as

$$\begin{pmatrix}11\\7\end{pmatrix} = 3\begin{pmatrix}3\\1\end{pmatrix} + 2\begin{pmatrix}1\\2\end{pmatrix}.$$

so $\mathcal{B}$-coordinates of $\begin{pmatrix}11\\7\end{pmatrix}$ are $c_1=3$

$c_2=2$.

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}3\\2\end{bmatrix}$$

if $\vec{v}=\begin{bmatrix}2\\3/2\end{bmatrix}$, it's $\frac{1}{2}\begin{bmatrix}3\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix}$

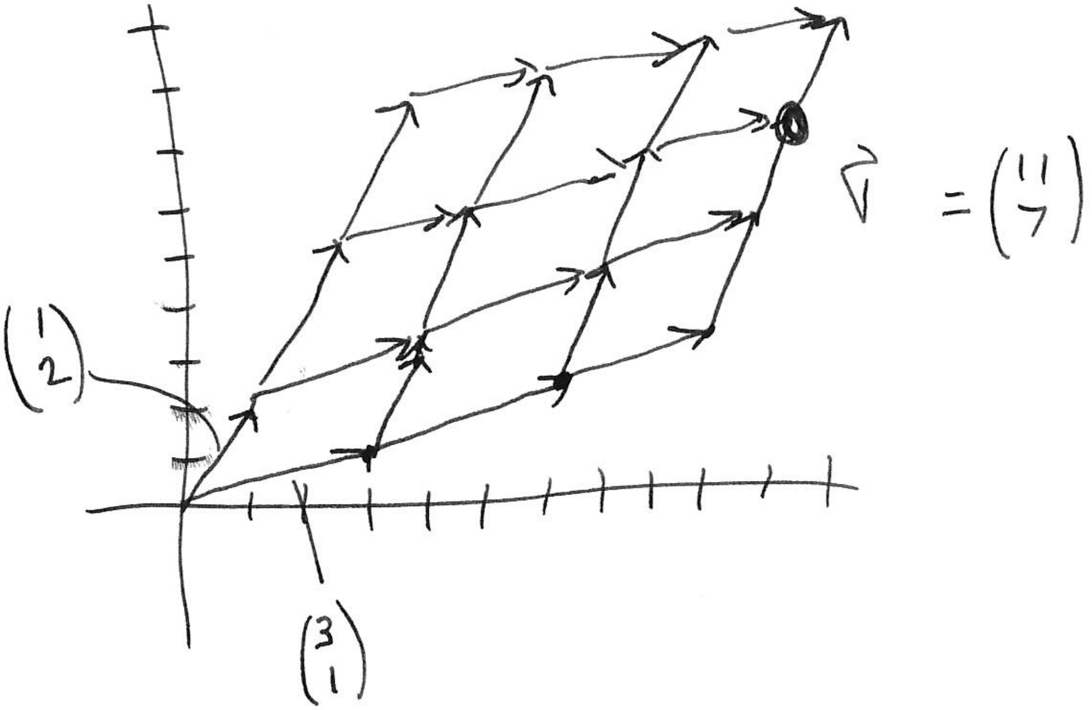
$c_1=1/2$

$c_2=1/2$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix}1/2\\1/2\end{bmatrix}$$

coordinates as "crooked graph paper"

for the basis $\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$



the vector with $\mathcal{B}$-coordinates $\begin{pmatrix}3\\2\end{pmatrix} = [\vec{v}]_{\mathcal{B}}$

is obtained by going 3 steps along $\vec{b}_1$,
2 steps along $\vec{b}_2$.

We've also talked about bases for other vector spaces.

e.g. $V = \mathbb{P}_2$ = polynomials of degree ≤ 2

$$at^2 + bt + c.$$

a basis for this vector space is

$\mathcal{B} = t^2, t, 1$

if we look at $\vec{p} = 3t^2 - 7t + 5$,

it's a combination:

$$(3)t^2 + (-7)t + (5).$$

the $\mathcal{B}$-coordinates of $\vec{p}$ are $\begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}$.

$$[\vec{p}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}.$$

$(0)t^2 + (3)t + (5)$

$\swarrow$

$\vec{p} = 3t + 5 \quad [\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

this is cool! we can check if polynomials are lin independent by checking if coordinate vectors are.

Next week

- I'll be traveling Tu, Th, F
- M, W quiz as usual, Skalit will sub F
- Replacement office hrs: M 10-11, W 10-11, or email me.
- This weekend: won't check my email F at 4 - Sunday afternoon; I'll respond to everything on Sunday.

subspace vs vector space

- a vector space is a set of things you can add and multiply by scalars.

- a subspace is a collection of vectors inside a fixed vector space such that's closed under addition, scaler mult.

example: vectors $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is subspace of $\mathbb{R}^2$.

source of confusion:
-(a subspace is also a vector space in its own right)

Last time:

if you have a vector space/subspace,

and a basis $\mathcal{B} = \vec{b}_1, \ldots, \vec{b}_n$.

any vector $\vec{x}$ can we written as combo of basis vectors:

$$\vec{x} = c_1 \vec{b}_1 + \cdots + c_n \vec{b}_n \text{ for some scalars } c_i .$$

this lets us encode the vector in coordinates:

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \text{ a vector in } \mathbb{R}^n.$$

the point: a) this makes it easier to check things in "weird" vector spaces. (e.g. is a set of polynomials lin. indep?)

b) can make life easier even in $\mathbb{R}^n$.

Suppose we have a basis for $\mathbb{R}^2$.

$$\mathcal{B} = \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}.$$

given a vector $\vec{x}$, how to find $[\vec{x}]_{\mathcal{B}}$?

this means: we want

these are the $\mathcal{B}$-coords for $\vec{x}$.

$$\vec{x} = c_1 \begin{pmatrix}3\\1\end{pmatrix} + c_2 \begin{pmatrix}1\\2\end{pmatrix}$$

e.g. $\begin{pmatrix}11\\7\end{pmatrix} = c_1 \begin{pmatrix}3\\1\end{pmatrix} + c_2 \begin{pmatrix}1\\2\end{pmatrix}$

what are c_1 & c_2?

$$\left[\begin{array}{cc|c} 3 & 1 & 11 \\ 1 & 2 & 7 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

$\begin{pmatrix}11\\7\end{pmatrix} = 3\begin{pmatrix}3\\1\end{pmatrix} + 2\begin{pmatrix}1\\2\end{pmatrix}$ $\quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix}3\\2\end{bmatrix}$

what if I tell you $[\vec{x}]_{\mathcal{B}}$ and $\mathcal{B}$ and ask what $\vec{x}$ is?

e.g. $\mathcal{B} = \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}3\\2\end{bmatrix}$.

what's $\vec{x}$?

well, it's $\vec{x} = 3\begin{bmatrix}3\\1\end{bmatrix} + 2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}11\\7\end{bmatrix}$.

ie. $\begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} = \vec{x}$.

this always works:

let $P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$

if I tell you $[\vec{x}]_{\mathcal{B}}$, then $\vec{x} = P_{\mathcal{B}}([\vec{x}]_{\mathcal{B}})$.

$P_{\mathcal{B}}$ = "change of basis matrix"

this always also useful the other way:

$[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x}$. lets you find $[\vec{x}]_{\mathcal{B}}$ given $\vec{x}$, like we did before.

The coordinate mapping is a linear transformation

$$\underset{\substack{\uparrow\\ \text{input:}\\ \text{in vector}\\ \text{space } V}}{\vec{x}} \longmapsto \underset{\substack{\uparrow\\ \text{output}\\ \text{in } \mathbb{R}^n}}{[\vec{x}]_{\mathcal{B}}}$$

it's linear transformation $T : V \to \mathbb{R}^n$.

it's both one-to-one and onto!

every vector in $\mathbb{R}^n$ corresponds to a unique vector in V.

this is called an isomorphism.

---

addition in the vector space corresponds to addition of coord vectors.

to check if some things in V are lin indep, check if the coordinate vectors are...

etc.

is $\{1-t^2, 1+t^2, t\}$ a basis for $\mathbb{P}_2$?

$\nwarrow$ quadratic polys

at^2+bt+c.

to check: use the basis $\overset{\underset{\shortparallel}{\mathcal{B}}}{1}, t, t^2$ for $\mathbb{P}_2$.

$$[1-t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad [1+t^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad [t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

equivalently, are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ a basis for $\mathbb{R}^3$?

check by row reduction:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

no free vars, so independent.
pivot in every row, so span.

$\Rightarrow$ the polynomials are a basis.

---

so any polynomial eg $3t^2+5t-7$, is a combination of those polynomials.

how is it a combination? what are c_i?

$$3t^2+5t-7 = c_1(1-t^2)+c_2(1+t^2)+c_3(t)$$

use coords to turn this
into a problem about vectors.

$$\Bigg\downarrow$$

$$c_1\begin{bmatrix}1\\0\\-1\end{bmatrix}+c_2\begin{bmatrix}1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3\\5\\-7\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&0&3\\0&0&1&5\\-1&1&0&-7\end{array}\right]\longrightarrow$$ row reduce, solve c_1, c_2, c_3.

---

e.g., for those who took diff. eq.:

maybe you know some diff eq has 2-dim'l solution space, and you know two solutions; check if they are indep by writing them in some basis for a corresponding space of functions

Important fact from 4.5:

if V is a vector space of subspace,

and a basis with n vectors in it.

$\Rightarrow$ any other basis also has n vectors.

the number of vectors in a basis is called the dimension of V.

eg. $\mathbb{R}^n$ has a basis $\vec{e}_1, \dots, \vec{e}_n$

so dimension of $\mathbb{R}^n$ is n.

$\mathbb{P}_2$ has a basis $1, t, t^2$.

dimension of $\mathbb{P}_2$ is 3

---

the dimension of a vectors space is the number of parameters needed to describe a vector.

Caveat: some vector spaces don't have a finite basis.

eg. $\mathbb{P}$ = all polynomials, any degree "infinite dimensional"

Main examples

if A is an $m \times n$ matrix.

$\dim(\text{Nul } A) =$ # of vectors in parametric vector form sol of $A\vec{x} = \vec{0}$.

$=$ #free variables when row reduce A.

$\dim((\text{Col } A) =$ # pivot columns

(since pivot cols are basis for col space)

- Office hrs this week: M 10-11 (or email me; some other time W)
  W 10-11

- HW 4.4.14 fixed in pdf

- Quiz W: 4.3, 4.4, 4.5

- Attendence sign-in today!

Using coordinates

in some weird vector space

if you have a vector space V, and a basis $\mathcal{B} = \vec{b}_1, \vec{b}_2, \vec{b}_3$

Want to know if $\vec{x}$ (in V) is a combo of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

What to do: write down $[\vec{x}]_\mathcal{B}$, $[\vec{v}_1]_\mathcal{B}$, $[\vec{v}_2]_\mathcal{B}$, $[\vec{v}_3]_\mathcal{B}$

actual vectors in $\mathbb{R}^3$.

check if $[\vec{x}]_\mathcal{B}$ is combo of $[\vec{v}_1]_\mathcal{B}, [\vec{v}_2]_\mathcal{B}, [\vec{v}_3]_\mathcal{B}$.

if it is, $\vec{x}$ is combo of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, <u>with same weights!</u>

Laguerre polynomials.

$\vec{v}_1(t) = 1$

$\vec{v}_2(t) = 1-t$

$\vec{v}_3(t) = 2-4t+t^2$

given the polynomial $\vec{x}(t) = t^2$.

write as combo of $\vec{p}_1, \vec{p}_2, \vec{p}_3$?

---

these are vectors in $\mathbb{P}_2$.

a basis for $\mathbb{P}_2$ is $1, t, t^2$.

---

in basis, want

$$c_1 \underset{\underset{[\vec{v}_1]_\mathcal{B}}{\uparrow}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + c_2 \underset{\underset{[\vec{v}_2]_\mathcal{B}}{\uparrow}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} + c_3 \underset{\underset{[\vec{v}_3]_\mathcal{B}}{\uparrow}}{\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}} = \underset{\underset{[\vec{x}]_\mathcal{B}}{\uparrow}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

this is just a linear system!

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{rref} \ldots} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$c_1 = 2$

$c_2 = -4$

$c_3 = 1.$

check: $2(1) + (-4)(1-t) + (1)(2-4t+t^2)$

$= 2 - 4 + 4t + 2 - 4t + t^2 = t^2 \checkmark$

check:

The row space of a matrix & rank

If A $m \times n$ matrix, then the row space of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Row $A = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}$

(in $\mathbb{R}^3$).

How to find it

Do row reduction on A until it's echelon form (don't need rref). call echelon form B.

A basis for the row space of A is given by nonzero rows of B. (not A)

Another way:

note: Row A is the same thing as $\text{Col}(A^T)$!

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

could then find basis using our method for column space.

row $A = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\} = \text{col } A^T$.

Example:

$A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$. find Row A, Col A, Nul A

(give a basis)

once we get A into rref form, can find all 3!

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\xrightarrow{\cdots}\begin{pmatrix}1&0&-1\\0&1&2\\0&0&0\end{pmatrix}.$$

$B=\text{rref}(A)$.

$$\text{Row } A=\text{span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$$ (nonzero rows of B)

$$\text{Col } A=\text{span}\left\{\begin{pmatrix}1\\4\\7\end{pmatrix},\begin{pmatrix}2\\5\\8\end{pmatrix}\right\}$$ (pivot cols of A)

$$\text{Nul } A=\text{span}\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=s\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

$$\uparrow$$

$$A\vec{x}=\vec{0}\rightsquigarrow\left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right]$$

$x_1=s$

$x_2=-2s$

$x_3=s$ is free

Notes

not rref

1. putting A in echelon form is enough to find
   Col A, Row A
2. for Nul A, need to use rref.
3. things in the basis of Col A are a bunch of cols of A
   the other ones are less similar.

Definition

the <u>rank</u> of A is the dimension of the column space of A.

Rank-nullity theorem (fundamental thm of linear algebra)

- $\dim \text{Row } A = \underset{\underset{\underset{\nwarrow \text{ rank}}{\text{rk } A}}{\shortparallel}}{\dim \text{Col } A}$ (always!)

- $\text{rk } A + \dim \text{Nul } A = \underset{\uparrow \text{ number of cols.}}{n}$ (where A is $m \times n$ matrix)

<u>Why?</u>
- row A has a basis vector for every pivot in echelon form (since every nonzero row has a pivot)
- col A has a basis vect for every pivot: basis is the pivot cols.

→ so dimension of each one is equal to number of pivots.

- nul A has a basis vector for each free variable.

the number of free variables = (number of cols) − (pivot columns)

$n - \text{rk } A$.

so $\dim \text{Nul } A + \text{rk } A = n$.

How to use it.

$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix}$.

what's dim Nul A? (what's dim of set of sols to $A\vec{x} = \vec{0}$)?

col $A = \text{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ ← all multiples of $\begin{pmatrix}1\\2\end{pmatrix}$.

so dim Col $A = 1$

dim Nul $A = 3$ ← #cols − dim Col A.

---

can a 6×9 matrix have 2-dimensional nullspace?

dim Nul A + dim Col A = 9

so if Nul A is 2-dim'l, Col A is 7-dim'l!

but Col A is a subspace of $\mathbb{R}^6$! (each col has six entries)

so dimension is <u>at most</u> 6, can't be 7.

so Nul A can't be 2-dim'l (must be at least 3-dim'l)

in other words... | a homogeneous system of 6 equations in 9 variables.

must have at least 3 dim'l set of solutions.

"Application"

A scientist has 40 eqns in 42 variables.

found two linearly indep solutions to $A\vec{x} = \vec{0}$, and there are no others.

does $A\vec{x} = \vec{b}$ have a solution for any $\vec{b}$?

---

dim Nul $A = 2$, since there are two independent sols.

theorem says: dim Col $A = 40$. $(42-2)$

so the columns span $\mathbb{R}^{40}$!

$\Rightarrow$ column space is $\mathbb{R}^{40}$.

$\Rightarrow$ $A\vec{x} = \vec{b}$ always has a sol.

Announcements.

- I'm gone for the rest of the week; no more office hours.
- Grades on ~~Piazza~~ Blackboard up to date (But it's not dropping the low quizzes)
- Change to schedule: §4.9 next, §5.1 on Monday.

(Re: a question)

Example:

basis: $\mathcal{B}$

$\vec{b}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ these are a basis for $\mathbb{R}^2$

$\vec{b}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

given $\vec{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, what is $[\vec{x}]_{\mathcal{B}}$?

want to get

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$ (solve for c_1, c_2)

find using row reduction: $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 3 & 4 \end{array}\right]$ tells you c_1, c_2.

$P_{\mathcal{B}}$

or $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}(\vec{x})$.

Change of basis

A given vector space has many possible bases.

Say we have two bases: $\mathcal{B} = \vec{b}_1, \vec{b}_2, \vec{b}_3$

$\mathcal{C} = \vec{c}_1, \vec{c}_2, \vec{c}_3$.

Basic question: given coordinates $[\vec{x}]_{\mathcal{B}}$,

how to find $[\vec{x}]_{\mathcal{C}}$?

First case: two bases for $\mathbb{R}^n$. $\mathcal{B} = \vec{b}_1, \dots, \vec{b}_n$

$[\vec{b}_1 \cdots \vec{b}_n]$ $\mathcal{C} = \vec{c}_1, \dots, \vec{c}_n$.

$\swarrow$

we know $\vec{x} = P_{\mathcal{B}}([\vec{x}]_{\mathcal{B}})$.

and $\vec{x} = P_{\mathcal{C}}([\vec{x}]_{\mathcal{C}})$.

$$P_{\mathcal{B}}([\vec{x}]_{\mathcal{B}}) = P_{\mathcal{C}}([\vec{x}]_{\mathcal{C}})$$

multiply by $P_{\mathcal{C}}^{-1}$ on left

$$(P_{\mathcal{C}}^{-1} P_{\mathcal{B}})([\vec{x}]_{\mathcal{B}}) = [\vec{x}]_{\mathcal{C}}.$$

great!

to get $[\vec{x}]_{\mathcal{C}}$, just multiply ~~on test~~ by some matrix.

this is called the change of basis matrix, written $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

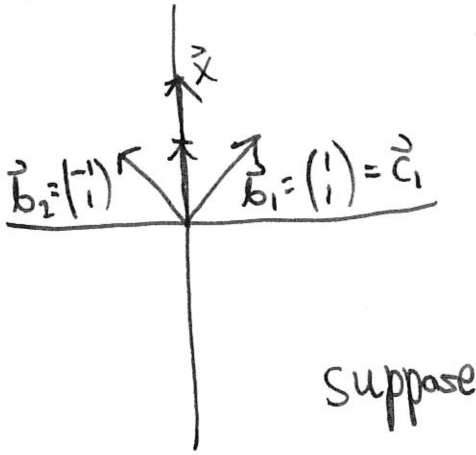
$C \leftarrow B$

for two bases in $\mathbb{R}^n$, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}$

(a little more complicated for other vector spaces;
if e.g. $P_{\mathcal{B}}$ is ~~change of~~ coord matrix for null(A),
it won't be square, and this doesn't work.)

it will still true that $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}([\vec{x}]_{\mathcal{B}})$, but
how to find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is different.

example:



two bases for $\mathbb{R}^2$:

$\mathcal{B}: \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$

$\mathcal{C}: \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

suppose $[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}1\\1\end{bmatrix}$. what's $[\vec{x}]_{\mathcal{C}}$?

$$P_{\mathcal{B}}=\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

$$P_{\mathcal{C}}=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}=P_{\mathcal{C}}^{-1}P_{\mathcal{B}}=\cdots=\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$$

$$[\vec{x}]_{\mathcal{C}}=\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$$

if $[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{x}=1\begin{bmatrix}1\\1\end{bmatrix}+1\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$

$[\vec{x}]_{\mathcal{C}}=\begin{bmatrix}0\\2\end{bmatrix}$, $\vec{x}=0\begin{bmatrix}1\\1\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}$ ✓.

Change of Basis

Let V be a finite dimensional vector space and suppose that $\mathcal{B}$ and $\mathcal{C}$ are bases for V.

For this discussion let $\dim V = 2$.

Say $\vec{x} \in V$ and $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

ie $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$. $\vec{x} = 5\vec{b}_1 + 2\vec{b}_2$.

We want to realize $[\vec{x}]_{\mathcal{C}}$ by multiplying $[\vec{x}]_{\mathcal{B}}$ by some matrix.

How do we find this matrix, $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$.

$$[\vec{x}]_{\mathcal{C}} = [5\vec{b}_1 + 2\vec{b}_2]_{\mathcal{C}} = 5[\vec{b}_1]_{\mathcal{C}} + 2[\vec{b}_2]_{\mathcal{C}}.$$

Thus, if put $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \left([\vec{b}_1]_{\mathcal{C}} \; [\vec{b}_2]_{\mathcal{C}} \right)$.

then $P_{\mathcal{C} \leftarrow \mathcal{B}} [\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{C}}$.

$$\left([\vec{b}_1]_{\mathcal{C}} \; [\vec{b}_2]_{\mathcal{C}} \right) \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5[b_1]_{\mathcal{C}} + 2[\vec{b}_2]_{\mathcal{C}} = [\vec{x}]_{\mathcal{C}}$$

Let V be the space of linear polynomials.

ie polynomials of the form $f(x) = ax + b$.

$\mathcal{B}$ consists of $\vec{b}_1(x) = 1+2x$

$\vec{b}_2(x) = 1-2x$

$\mathcal{C}$ consists of $\vec{c}_1(x) = 1+x$

$\vec{c}_2(x) = 1-x$

$[\vec{b}_1]_{\mathcal{C}} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$ $\quad \frac{3}{2}(1+x) - \frac{1}{2}(1-x)$

$\frac{3}{2} + \frac{3}{2}x - \frac{1}{2} + \frac{1}{2}x = 1+2x$

$= \vec{b}_1(x).$

$(\vec{b}_2]_{\mathcal{C}} = \begin{pmatrix} -1/2 \\ 3/2 \end{pmatrix}.$

$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$

Let's consider $[f]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ ie $f(x) = 2(1+2x)+4(1-2x)$

$= 2+4x+4-8x$

$= \underline{6-4x}$

What is $[f]_{\mathcal{C}}$?

$[f]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\vec{f}]_{\mathcal{B}} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

Thus $f(x) = 1 \cdot \vec{c}_1(x) + 5 \cdot \vec{c}_2(x)$

$= (1+x) + 5(1-x).$

$= 1+x+5-5x$

$= 6-4x$ ✓

Markov Chains:

Example: Each year 5% of the people in the city move to the suburbs. (95% stay)

3% of the people in the suburbs move to city (97% stay put).

$$\vec{x}_0 = \begin{pmatrix} c \\ s \end{pmatrix} \begin{matrix} \leftarrow \text{\# of people in the city} \\ \leftarrow \text{\# of people in the suburbs.} \end{matrix}$$

$$\vec{x}_1 = \begin{pmatrix} \quad \\ \quad \end{pmatrix} \begin{matrix} \leftarrow \text{\# of people in city after one year} \\ \leftarrow \text{\# of people in the suburbs after one year.} \end{matrix}$$

$$A\vec{x}_0 = \vec{x}_1$$

$$A = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}$$

$$A\vec{x}_0 = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} = \begin{pmatrix} 0.95c + 0.03s \\ 0.05c + 0.97s \end{pmatrix}.$$

In our situation we are going to assume that $c + s = 1$. (ie 1 represents 100% of the total population and c, s are the proportions living in the city / suburbs respectively).

Idea: we can see the long term growth of this system because

$\vec{x_n} = A\,\vec{x_{n-1}}$ where $\vec{x}_n$ is proportions of people living in the city / suburbs after n years.

Note: $\vec{x}_1 = A\vec{x}_0$

$\vec{x}_2 = A\vec{x}_1 = A(A\vec{x}_0) = A^2\vec{x}_0$

$\vec{x}_3 = A\vec{x}_2 = A(A^2\vec{x}_0) = A^3\vec{x}_0$

$\vdots$

$\vec{x_n} = A^n\vec{x}_0.$

Want to know $\lim_{n\to\infty} \vec{x}_n$.

Definitions: A probability vector is some $\vec{x} \in \mathbb{R}^n$ where $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ w/ $x_1, x_2, \dots x_n \geq 0$.

$x_1 + x_2 + \dots + x_n = 1.$

A stochastic nxn matrix A is a matrix whose columns are probability vectors.

Fact: If A is stochastic and $\vec{x}$ is a probability vector then $A\vec{x}$ is a probability vector.

A markov chain is a sequence $(\vec{x}_n)$ where

$\vec{x}_n = A\vec{x}_{n-1}$ and $\vec{x}_0$ is probability vector

A is stochastic.

In our example.

$$\vec{x}_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \begin{matrix} \leftarrow \text{people in city} \\ \leftarrow \text{people in suburbs.} \end{matrix}$$

(probability vector) initial condition

$$A = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}$$ stochastic matrix.

$\vec{x}_n = A\vec{x}_{n-1}$ so $(\vec{x}_n)$ is a Markov chain.

Know that $\vec{x}_n = A^n \vec{x}_0$.

A priori, requires a lot of computation)

In our experiment, $\vec{x}_n$ seems to be converging to $\begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix} = \vec{q}$.

One can also see that

$$A\vec{q} = \vec{q}$$

$\vec{q}$ is called a steady state. Question: how do we find steady states without doing lots of computation?

We want to find a steady state probability vector $\vec{q}$ for A.

We find it by considering the equation

$$A\vec{q} = \vec{q} \iff A\vec{q} - \vec{q} = \vec{0}$$
$$\iff A\vec{q} - I\vec{q} = 0$$
$$\iff (A - I)\vec{q} = 0.$$

We see that a steady state vector $\vec{q}$ is a vector in the nullspace of $A - I$.

__Thm__: If A is stochastic, then it admits at least one steady state vector.

Look at $A - I = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{pmatrix}.$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & -3/5 \\ 0 & 0 \end{pmatrix}$$

$$\text{Nul}(A - I) = \left\{ t\begin{pmatrix} 3/5 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$
$$= \text{span}\left\{ \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} \right\}.$$

The steady state is a probability vector, so the entries must sum to 1, so look at

$$\begin{pmatrix} 3/5 \\ 1 \end{pmatrix} \cdot \frac{1}{1 + 3/5} = \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} \cdot (5/8) = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix}$$

which is the experimentally found steady state!

Question: Can there exist more than one steady state?

Answer: Yes (stupid example).

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ A is stochastic.

but for any probability vector $\vec{p}$.

$A\vec{p} = \vec{p}$.

~~Thm~~: Def: We call a stochastic matrix A regular if A^k has no nonzero entries for some k.

Thm: If A is a regular stochastic matrix, then A admits a unique steady state.

And for any initial vector $\vec{x}_0$ Markov chain $\vec{x}_n = A\vec{x}_{n-1}$

$\lim_{n\to\infty} \vec{x}_n$ converges to this steady state. //

Announcements

- Quiz W: 4.6, 4.7, 4.9
- Normal OH this week
- Start §5.

- If V is a vector space or subspace,

  (any basis) ↓

  the dimension of V is number of vectors in a basis

- How to compute it?

  → find a basis for V (by appropriate method), count vectors in it.

- = idea: how many ~~vecto~~ parameters are needed to describe an element of V?

Change of basis.

Here are two bases for $\mathbb{R}^2$.

$$\mathcal{B} = \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix};$$

$$\mathcal{C} = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}.$$

idea: $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{x}]_{\mathcal{B}}$ just apply a matrix! to switch coords

For two bases for $\mathbb{R}^n$,

$$\underline{\underline{\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}}}$$

where $P_{\mathcal{B}} = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$

$$P_{\mathcal{C}} = \begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}$$

$$P_{\mathcal{C}}^{-1} = \frac{1}{-1}\begin{bmatrix}-1 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}$$

not usually $= P_{\mathcal{C}}^{-1}$; that's what I get for making up an example on the fly.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}0 & -1\\-1 & -2\end{bmatrix}.$$

suppose $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

then $[\vec{x}]_{\mathcal{E}} = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \end{bmatrix}$

check: $\vec{x} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ (find $\vec{x}$ using $[\vec{x}]_{\mathcal{B}}$)

$\vec{x} = (-3) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-7) \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ (find $\vec{x}$ using $[\vec{x}]_{\mathcal{E}}$)

same thing, phew.

Markov example

Suppose we have a chemical with two different isomers:

A: very unstable

B: very stable

simple model: every second, 90% of A turns into B

10% stays A

5% of B turns to A

95% of B stays B.

> in reality, it's a continuous process.

$$M = \begin{bmatrix} 0.1 & 0.05 \\ 0.9 & 0.95 \end{bmatrix}. \qquad \vec{X}_{k+1} = M \underset{\begin{bmatrix} \% A \\ \% B \end{bmatrix}}{\underset{\nearrow}{\vec{x}_k}}.$$

---

in the long run, reaches some equilibrium. that's what the steady state is. it should solve $M\vec{x} = \vec{x}$.

so $(M - I)\vec{x} = \vec{0}$. to find $\vec{x}$: write down $M - I$,

solve $(M - I)\vec{x} = \vec{0}$

using row red.

$$M-I=\begin{bmatrix}0.1 & 0.05\\ 0.9 & 0.95\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-0.9 & 0.05\\ 0.9 & -0.05\end{bmatrix}$$

$(M-I)\vec{x}=\vec{0}$ gives $\vec{x}=\begin{bmatrix}1\\18\end{bmatrix}$ ← 18 parts B per 1 part A.

---

if $\vec{x}$ is a sol of $M\vec{x}=\vec{x}$, any multiple of $\vec{x}$ is too.

$\vec{x}=\begin{bmatrix}1/19\\ 18/19\end{bmatrix}$ is a state vector that's equilibrium.

└ add up to 1.

(just divide by total)

Note: if you solve $M\vec{x}=\vec{x}$ and $\vec{x}$ has negative entries, you messed up.

(at least if M stochastic)

§5.1: Eigenstuff, pt 1.

if A is an $n \times n$ matrix, $\vec{x}$ a vector, then

$A\vec{x}$ is another vector, which isn't much like $\vec{x}$ most of the time.

$\vec{x}$ is an eigenvector of A means that

$\boxed{A\vec{x} = \lambda\vec{x}}$ ← just a rescaled version of $\vec{x}$; same direction!

↑ a scalar; called the eigenvalue of $\vec{x}$.

Example $\quad A = \begin{pmatrix} 7 & -4 \\ -2 & 5 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$A\vec{x} = \begin{pmatrix} 7 & -4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A, with eigenvalue 3.

most vectors aren't eigenvectors of a given matrix.

two different eigenvectors of a single matrix typically have different eigenvalues.

How to find the eigenvectors?

~~if~~ suppose λ is a scalar, and you want to know the eigenvects with eigenval λ.

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} = (\lambda I)\vec{x}$$

$$A\vec{x} - (\lambda I)\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}.$$

so: compute $A - \lambda I$, solve $(A - \lambda I)\vec{x} = \vec{0}$ for $\vec{x}$.

let's do last example in reverse.

$A = \begin{pmatrix} 7 & -4 \\ -2 & 5 \end{pmatrix}$, $\lambda = 3$. Find the eigenvectors with eigenval 3 ("3-eigenvector")

$$A - \lambda I = \begin{pmatrix} 7 & -4 \\ -2 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix}.$$

$$(A - \lambda I)\vec{x} = \vec{0} \rightsquigarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ -2 & 2 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note: the set of all λ-eigenvectors is a subspace of $\mathbb{R}^n$.

(it's all sols to $(A-\lambda I)\vec{x}=\vec{0}$; $\mathrm{Nul}\,(A-\lambda I)$).

in example, it's 1-dimensional, but that's not always true.

---

some notes:

- for most λ's, there are no eigenvectors!

  e.g. $A=\begin{pmatrix}7&-4\\-2&5\end{pmatrix}$, $\lambda=5$.

  ~~$A-\lambda$~~ $(A-\lambda I)=\begin{pmatrix}7&-4\\-2&5\end{pmatrix}-\begin{pmatrix}5&0\\0&5\end{pmatrix}=\begin{pmatrix}2&-4\\-2&0\end{pmatrix}$

  ↗ invertible, so $\vec{x}=\vec{0}$ is only sol.
  (doesn't really count)

- next time: how to figure out which λ's actually have eigenvectors?

- what's a 1-eigenvector?
  $A\vec{x}=1\cdot\vec{x}=\vec{x}$ (e.g. steady state for Markov process)

- what's a 0-eigenvector? $A\vec{x}=0\cdot\vec{x}=\vec{0}$. ($\vec{x}$ is in Nul A)

Applying $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$:

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is evector, with $\lambda=1$.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is evector, with $\lambda=2$.

$\vec{x}$ is an eigenvector if $A\vec{x}$ is parallel to $\vec{x}$.

e.g $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ isn't.

Applying $\begin{bmatrix} 1 & 0 \\ -0.47 & 1 \end{bmatrix}$:

$\binom{0}{1}$ is an ↑ evector; there are no others except multiples of this.

Announcements

- Quizzes not all graded yet. Will return F - sorry!
- Midterm in two weeks: stay tuned for details.
  Let me know ASAP if you need a make-up.
- Today: finding eigenvalues / eigenvectors.

How to find eigenvalues/eigenvectors?

A an $n \times n$ matrix

Remember; eigenvector $\vec{x}$ is vector so $A\vec{x} = \lambda\vec{x}$

λ called eigenvalue.

Last time: given A, given λ, find all eigenvectors with eigenvalue λ by finding $\text{Nul}(A - \lambda I)$.

if $\vec{x}$ in nullspace, then $(A - \lambda I)\vec{x} = \vec{0}$

$$A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$A\vec{x} = \lambda\vec{x} \Longrightarrow \vec{x} \text{ eigenvector.}$$

But! for most λ, there are no eigenvectors (other than $\vec{0}$)

how to find the λ's that work?

want: find λ so $A - \lambda I$ has nonzero nullspace.

$\rightarrow$ so find λ that make $\det(A - \lambda I) = 0$.

just compute $\det(A - \lambda I)$, in terms of λ.

$\det(A - \lambda I)$ is polynomial in terms of λ called "characteristic polynomial"

roots of this polynomial are the eigenvalues.

ex

$$A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$$

characteristic poly is

$$\det(A-\lambda I)=\det\left(\begin{pmatrix}1&2\\2&4\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\right)$$

$$=\det\begin{pmatrix}1-\lambda&2\\2&4-\lambda\end{pmatrix}$$

$$=(1-\lambda)(4-\lambda)-(2)(2)=4-5\lambda+\lambda^2-4$$

$$=\lambda^2-5\lambda=\lambda(\lambda-5).$$

roots are $\lambda=0$
$\lambda=5$ ← these are eigenvalues.

now we can find corresponding eigenvectors, as on Monday.

do each eigenval separately.

$\lambda=0$: $A-\lambda I=A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$

want $(A-\lambda I)\vec{x}=0$.

$\rightsquigarrow \vec{x}=\begin{pmatrix}-2\\1\end{pmatrix}$. check:

$\begin{pmatrix}1&2\\2&4\end{pmatrix}\begin{pmatrix}-2\\+1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=0\cdot\vec{x}$ ✓

$\lambda=5$. $A-\lambda I=\begin{pmatrix}1&2\\2&4\end{pmatrix}-\begin{pmatrix}5&0\\0&5\end{pmatrix}$

$=\begin{pmatrix}-4&2\\2&-1\end{pmatrix}$.

$(A-\lambda I)\vec{x}=\vec{0} \rightsquigarrow \begin{pmatrix}1\\2\end{pmatrix}=\vec{x}$.

check: $A\vec{x}=\begin{pmatrix}1&2\\2&4\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}5\\10\end{pmatrix}$

~~$(A-\lambda I)\vec{x}$~~

$=5\vec{x}$ ✓

If A is triangular, life is easy:

$$A = \begin{pmatrix} 1 & 3 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

char. poly:

$$\det(A-\lambda I) = \det\left[\begin{pmatrix} 1 & 3 & 7 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right]$$

$$= \det\begin{pmatrix} 1-\lambda & 3 & 7 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

determinant is $\underset{\substack{\nearrow \\ \text{char poly.}}}{(1-\lambda)(2-\lambda)(3-\lambda)}$ (multiply diagonal entries to get det)

roots are $\lambda = 1, 2, 3$.

for any triangular matrix, the eigenvalues are just the numbers on the diagonal. then find eigenvectors as before.

For a non-triangular matrix, eigenvalues will have square roots, fractions, and worse!

Important fact:

if A is an $n \times n$ matrix,

and $\vec{v}_1, \dots, \vec{v}_r$ are eigenvectors, ~~the~~
with different eigenvalues,

then $\vec{v}_1, \dots, \vec{v}_r$ are linearly independent.

most $n \times n$ matrices have n different eigenvalues.

if you take one $\vec{v}_i$ for each λ_i, you get a basis:
"eigenbasis"

(Didn't do a 3×3 one in class; including here for reference)

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & -6 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & -6-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{pmatrix}$$

$$\det(A-\lambda I) = (7-\lambda)\det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 6-\lambda \end{pmatrix}$$

$$= (7-\lambda)\Big((2-\lambda)(6-\lambda)-9\Big)$$

$$= (7-\lambda)(-21+4\lambda+\lambda^2)$$

$$= (7-\lambda)(\lambda-3)(\lambda+7)$$

the eigenvalues are $\lambda = -7, 3, 7$.

for each one, find an eigenvector by solving

$$(A-\lambda I)\vec{x} = \vec{0}.$$

Announements

- Quizzes back M - sorry again!
- Exam details, practice to be posted M.

(2.5 - 5.3)
″ ″
LU today

Eigenvectors/Values for 3x3 matrices

$$A=\begin{pmatrix}2 & 3 & 1\\ 3 & -6 & 2\\ 0 & 0 & 7\end{pmatrix}$$

1. figure out characteristic polynomial.

$$\det(A-\lambda I)=\begin{pmatrix}2-\lambda & 3 & 1\\ 3 & -6-\lambda & 2\\ 0 & 0 & 7-\lambda\end{pmatrix}$$

use cofactor expansion, 3rd row

$$=0(\sim)-0(\sim)+(7-\lambda)\det\begin{pmatrix}2-\lambda & 3\\ 3 & -6-\lambda\end{pmatrix}$$

$$=(7-\lambda)\left[(2-\lambda)(-6-\lambda)-9\right]$$

$$=(7-\lambda)(-21+4\lambda+\lambda^2)$$

$$=(7-\lambda)(\lambda-3)(\lambda+7)$$

(if you don't see how to factor, use quadratic formula)

for general $n\times n$ matrix, get a degree n poly in λ, which is hard.

eigenvalues: 7, 3, -7.

2. find the eigenvectors for each eigenvalue.

$\lambda = -7.$

$$A - \lambda I = \begin{pmatrix} 9 & 3 & 1 \\ 3 & 1 & 2 \\ 0 & 0 & 14 \end{pmatrix}$$

solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\left(\begin{array}{ccc|c} 9 & 3 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 0 & 14 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 9 & 3 & 1 & 0 \\ 0 & 0 & 14 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 14 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 14 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

solution: $x_1 = -3x_2$ $\quad 3x_1 + x_2 = 0$

x_2 is free $\quad x_1 = -\frac{1}{3}x_2$

$x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \cancel{\begin{pmatrix} -3s \\ s \\ 0 \end{pmatrix}} = \cancel{\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} s}.$$

$$= \qquad = \begin{pmatrix} -1/3 \\ 1 \\ 0 \end{pmatrix} s$$

↖ eigenvector.

Tip: quick way for 2×2 eigenvectors.

$A=\begin{pmatrix} 4 & 3 \\ 4 & 8 \end{pmatrix}$ if you compute char poly, you'll find that $\lambda=2$ is an eigenval.

$A-2I=\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}.$

$(A-2I)\vec{x}=0 \rightsquigarrow \left(\begin{array}{cc|c} 2 & 3 & 0 \\ 4 & 6 & 0 \end{array}\right) \rightarrow \cdots$

to get $\vec{x}$, switch the entries of first row, add - sign to first one.

$\vec{x}=\begin{pmatrix} -3 \\ 2 \end{pmatrix}.$

$\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (same answer row red gives)

works because
(it's 2x2 and we know there's a nullspace.)

A problem we've run into a couple times:

given A, compute A^{100}.

---

if A is diagonal, it's easy ↙ all 0 except on main diagonal

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^3 & 0 \\ 0 & 3^3 \end{pmatrix}.$$

...

$$A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}$$

Similarly easy if $A = \underset{\substack{\uparrow \\ \text{any matrix} \\ \text{invertible}}}{P} \underset{\nwarrow \text{diagonal}}{D} P^{-1}$

$$A^{100} = \underbrace{(PDP^{-1})(PDP^{-1})\cdots\cdots(PDP^{-1})}_{100 \text{ times}}$$

$$= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)D\cdots\cdots P^{-1}$$

$$= PDIDIDI\cdots P^{-1}$$

$$= PD^{100}P^{-1}$$

but this is easy!

- D diagonal, so D^{100} is just raising diagonals too 100^{th} power.
- left with $PD^{100}P^{-1}$: three matrices only!

e.g.

$$A = \underset{\substack{\uparrow \\ P}}{\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}} \underset{\substack{\uparrow \\ D}}{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}} \underset{\substack{\uparrow \\ P^{-1}}}{\begin{pmatrix} -1 & +1 \\ +2 & -1 \end{pmatrix}} = \begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}.$$

$$A^{100} = PD^{100}P^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 \\ 0 & 3^{100} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ +2 & -1 \end{pmatrix}$$

Important fact

almost every matrix can be written as $A = PDP^{-1}$,

for suitable P & D!

this is called "diagonalization" of A.

Here's how to find P & D:

1. find the eigenvalues, by finding roots of char poly.
   (usually are all different)
2. find an eigenvector for each eigenvalue (get n of them)
3. form P using eigenvectors as columns
4. form D by putting eigenvals on diagonal (in same order!)

Warning: if there's a repeated eigenvalue, this won't always work.

example. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ can't be diagonalized.

(Google "Jordan canonical form" if you run into one.)

Example:

$$A=\begin{pmatrix} 2 & 3 & 1 \\ 3 & -6 & 2 \\ 0 & 0 & 7 \end{pmatrix}$$

$$\begin{array}{c} P=\begin{pmatrix} -1/3 & 19 & 3 \\ 1 & 13 & 1 \\ 0 & 56 & 0 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \lambda=-7 & \lambda=7 & \lambda=3. \end{matrix} \end{array}$$

$$D=\begin{pmatrix} -7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

check: $A=PDP^{-1}$, can compute A^{100} very fast.

Many uses other than A^{100}!

diagonalization is very important!

Announcements

- Exam next W: 2.5-5.3

  practice test posted; review sheet soon.

- Review sessions M, T late afternoon, TBA

- Quiz ~~tomorrow~~ Wednesday, 5.1-5.3

The deal with repeated eigenvalues.

sometimes an eigenvalue shows up more than once.

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}. \quad \lambda = 2 \text{ and } 2.$$

when this happens, there can be several linearly indep eigenvectors. but this doesn't always happen.

e.g. $$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \lambda = 2 \text{ and } 2.$$

to find the eigenvectors with $\lambda = 2$ (2-eigenspace).

$$(B - \lambda I) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } \lambda = 2.$$

2-eigenspace = $\mathrm{Nul}(B - \lambda I)$ = all of $\mathbb{R}^2$

a basis: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$\lambda = 2$ showed up twice, and there are 2 linearly indep eigenvectors.

example 2

$$A=\begin{pmatrix}2&1\\0&2\end{pmatrix} \qquad \lambda=2,2.$$

$$\text{2-eigenspace}=\operatorname{Nul}(A-\lambda I)=\operatorname{Nul}\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\right).$$

$$\left(\begin{array}{cc|c}0&1&0\\0&0&0\end{array}\right). \quad \text{already rref!}$$

$$\begin{array}{l}x_1 \text{ free}\\ x_2=0.\end{array} \rightsquigarrow \begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}s\\0\end{pmatrix}=s\begin{pmatrix}1\\0\end{pmatrix}.$$

$\begin{pmatrix}1\\0\end{pmatrix}$ is eigenvector, but no others!

$\lambda=2$ showed up twice,
but there's only 1 linearly
indep eigenvector

---

in general:

#linearly indep eigenv~~alues~~ectors w/ eigenvalue λ $\leq$ # of times λ shows up

↑

dim Nul$(A-\lambda I)$

but not always equal!

so an nxn matrix might have fewer than n linearly indep eigenvectors, if repeated eigenvalues.

---

if you try to diagonalize a matrix with < n eigenvectors, when you try to form P, you don't have enough columns.

P won't be square

if < n eigenvectors, A can't be diagonalized.

eg. $A=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

if A and B are n×n matrices, and $A = PBP^{-1}$

for some P. then A and B are similar.

---

e.g if $A = PDP^{-1}$ is diagonalization, A&D are similar.

- if A&B are similar, $\det(A) = \det(B)$

why? $\det(B) = \det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1})$

$= \det(P)\det(P^{-1})\det(A)$

$= \det(A)$.

- if A&B are similar, they have the same eigenvalues.

- $\det(A) = \det(D)$ if $A = PDP^{-1}$

$$= \det\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \lambda_1 \lambda_2 \lambda_3.$$

so $\det(A)$ = product of eigenvals.

§5.4 Linear transformations in different coord systems. (+ a use for diagonalization!)

---

say we have a linear trans.

$T: \mathbb{R}^n \to \mathbb{R}^m$. (given by $m \times n$ matrix, probably)

$\mathcal{B}$ basis for $\mathbb{R}^n$

$\mathcal{C}$ basis for $\mathbb{R}^m$

→ there's a matrix M so that

$$[T(\vec{x})]_{\mathcal{C}} = M[\vec{x}]_{\mathcal{B}}$$

if you know $[\vec{x}]_{\mathcal{B}}$, and want to know $[T(\vec{x})]_{\mathcal{C}}$, do $M[\vec{x}]_{\mathcal{B}}$.

how to find it:

$$M = [T(\vec{b}_1)_{\mathcal{C}} \cdots T(\vec{b}_n)_{\mathcal{C}}]$$

take $\vec{b}_1$. find $T(\vec{b}_1)$, and then work out $[T(\vec{b}_1)]_{\mathcal{C}}$. that's first column.

repeat for other $\vec{b}_j$'s.

$T : \mathbb{R}^2 \to \mathbb{R}^2$

given by $A = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}$.

use $\mathcal{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as first basis

$\mathcal{C} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

to find M:

$T(\vec{b}_1) = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $[T(\vec{b}_1)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, since

$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{b}$

$T(\vec{b}_2) = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $[T(\vec{b}_2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, since

$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

---

check: $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $T(\vec{x}) = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $M[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

is $[T(\vec{x})]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$? $4\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 7 \end{bmatrix}$ ✓

so $\left(\begin{bmatrix} 11 \\ 7 \end{bmatrix}\right)_{\mathcal{C}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

what if m=n, so we have $T:\mathbb{R}^n \to \mathbb{R}^n$?

also use eigenbasis $\mathcal{B}$ for both $\mathbb{R}^n$'s.

---

if T given by matrix A and $A=PDP^{-1}$,

then the matrix for T is $\mathcal{B}$-coords is just diagonal matrix

D!

---

what if A has $\lambda_1=3$, $\vec{v}_1=\begin{pmatrix}1\\2\end{pmatrix}$,

$\lambda_2=4$, $\vec{v}_2=\begin{pmatrix}3\\1\end{pmatrix}$.

if $\vec{x}=2\begin{pmatrix}1\\2\end{pmatrix}-1\begin{pmatrix}3\\1\end{pmatrix}$, and want $A\vec{x}$.

$$A\vec{x}=2A\begin{pmatrix}1\\2\end{pmatrix}-1A\begin{pmatrix}3\\1\end{pmatrix}=2\cdot 3\cdot\begin{pmatrix}1\\2\end{pmatrix}-1\cdot 4\cdot\begin{pmatrix}3\\1\end{pmatrix}$$

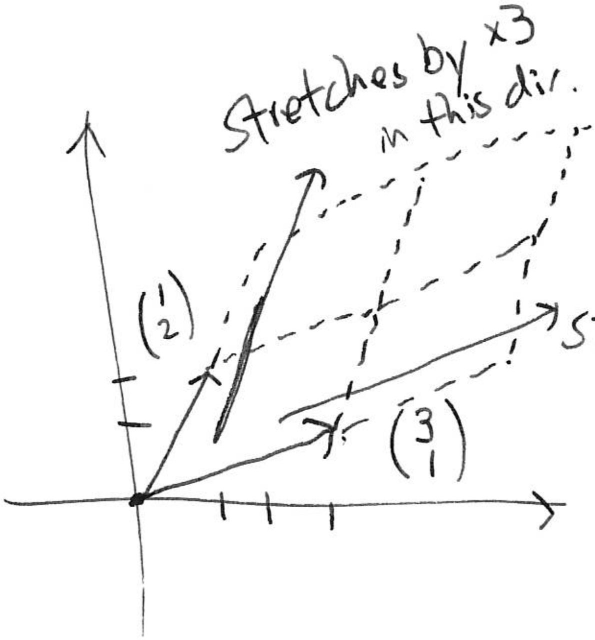
$$=6\begin{pmatrix}1\\2\end{pmatrix}-4\begin{pmatrix}3\\1\end{pmatrix}.$$

$[x]_{\mathcal{B}}=\begin{bmatrix}2\\-1\end{bmatrix}$, $[Ax]_{\mathcal{B}}=\begin{bmatrix}6\\-4\end{bmatrix}$.

so $[Ax]_{\mathcal{B}}=\begin{bmatrix}3&0\\0&4\end{bmatrix}[\vec{X}]_{\mathcal{B}}$

- if you know $\vec{x}$ as combo of eigenvects, get $A\vec{x}$ as combo of eigenvects by applying D.

The picture: $T: \mathbb{R}^2 \to \mathbb{R}^2$



how does T act on crooked graph paper?

Announcements

Q next

- Exam next W
- Quizzes will be returned M
- Pick up Quiz 8 on the way out today
- Tell me if you need a make-up!

- Quick review of complex numbers: Appendix B.

How to diagonalize a matrix A

$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$

1. find the eigenvalues.

want: $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 0 \cdot 2$$

$$= (1-\lambda)(3-\lambda).$$

roots are $\lambda = 1$, $\lambda = 3$.

2. Find an eigenvector for each eigenvalue.

for each λ, solve $(A - \lambda I)\vec{x} = \vec{0}$.

$\lambda = 1$: $A - \lambda I = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$. $(A - \lambda I)\vec{x} = \vec{0} \rightsquigarrow \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 2 & 0 \end{array}\right]$

row reduction.

$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\lambda = 3$: $A - \lambda I = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$. $(A - \lambda I)\vec{x} = \vec{0}$.

$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3. write down P & D:

$$P = \overset{\substack{\text{evectrs}\\ \downarrow \qquad \searrow}}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \qquad D = \overset{\substack{\text{eigenvals}\\ \searrow}}{\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}.$$

$$\boxed{A = PDP^{-1}}$$

why eigenbases useful?

↑

for most matrices A, the eigenvectors give a basis for $\mathbb{R}^n$ (where A is $n\times n$)

that's an eigenbasis.

say we want to know $A^n\vec{x}$ for large n.

if $\vec{x}$ is a combination of eigenvectors, this is easy:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \qquad \vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$A\vec{x} = 1\cdot 1\cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\cdot 3\cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^2\vec{x} = 1\cdot 1^2\cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\cdot 3^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^3\vec{x} = 1\cdot 1^3\cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\cdot 3^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\dots,$$

$$A^n\vec{x} = 1\cdot 1^n\cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\cdot 3^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

in the formulas from Monday,

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}1\\2\end{bmatrix} \quad \left(\text{since } \vec{x} = 1\overset{\vec{b}_1}{\begin{bmatrix}1\\0\end{bmatrix}} + 2\overset{\vec{b}_2}{\begin{bmatrix}1\\1\end{bmatrix}}\right)$$

$$\cancel{A^n\vec{x} = \begin{bmatrix}1^n & 0\\0 & 3^n\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}}$$

$$[A^n\vec{x}]_{\mathcal{B}} = \begin{bmatrix}1^n & 0\\0 & 3^n\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\cdot 1^n\\2\cdot 3^n\end{bmatrix}$$

$$\text{since } A^n\vec{x} = 1\cdot 1^n\begin{bmatrix}1\\0\end{bmatrix} + 2\cdot 3^n\begin{bmatrix}1\\1\end{bmatrix}$$

this shows:

$$\boxed{[A^n\vec{x}]_{\mathcal{B}} = D^n\,[\vec{x}]_{\mathcal{B}}.}$$

(key formula from Monday!)

looks scary, but it's just a compact way to write the observations from the first page.

Recap of complex numbers

What are eigenvalues of

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}?$$ (rotation by 90°)

$$\det(A-\lambda I)=\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}=\lambda^2+1.$$

so $\lambda^2+1=0$.

uhoh, - need complex numbers: $\lambda=i$ or $-i$.

make sure you know how to:

- add complex #s
- multiply them
- divide
- work in polar coords: $a+bi=re^{i\theta}$.

(this is explained well in Appendix B)

(or many many places online; email me to set up a time to go over it.)

Announcements

- Review sessions M, 4-5
  Tu, 5-6 in SEO 612

  (go one practice test) (2.5 - 5.3)

- I'll post practice sols this weekend.

- Today: dealing with complex eigenvalues.

- Attendence check!

A problem:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

char poly: $\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2+1.$

eigenvalues: i & $-i$.

- Could diagonalite as before, but P&D will be complex!
- this would be bad, eg in physics problems.
- how to interpret this?

What to do

Find the eigenvectors, with complex entries, by usual method.

eg. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\lambda = i$.

$A - iI = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$

solve $(A - iI)\vec{x} = 0$, using row red.

$$\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{R1 \times i} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow[\text{to } R2]{\text{add } -R1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 - i x_2 = 0$

$x_1 = i x_2$
$x_2 =$ free plug in $x_2 = 1 \Rightarrow \vec{x} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

~~do~~ check: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} = i\begin{pmatrix} i \\ 1 \end{pmatrix}$.

note: scalar × eigenvector is also eigenvector

$i\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix}$ is also an eigenvector with $\lambda = i$.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = \lambda^2 + \lambda + 1.$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}.$$

$$\lambda = \frac{-1-\sqrt{3}i}{2}: \quad A - \lambda I = \begin{pmatrix} \frac{1+\sqrt{3}i}{2} & 1 \\ -1 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}$$

$$\text{eigenvector: } \begin{pmatrix} -1 \\ \frac{1+\sqrt{3}i}{2} \end{pmatrix}.$$

$$\lambda = \frac{-1+\sqrt{3}i}{2}: \quad A - \lambda I = \begin{pmatrix} \frac{1-\sqrt{3}i}{2} & 1 \\ -1 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix}.$$

$$\text{eigenvector: } \begin{pmatrix} -1 \\ \frac{1-\sqrt{3}i}{2} \end{pmatrix}.$$

if $z = a+bi$, the complex conjugate is $a-bi$. this is written $\overline{z}$.

you probably noticed above:

if λ is an eigenvalue of A, so is $\overline{\lambda}$.

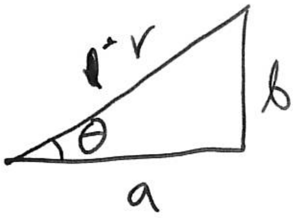
if $\vec{v}$ is eigenvector with eigenvalue λ, then $\overline{\vec{v}}$ is an eigenvector with eigenvalue $\overline{\lambda}$.

so if you find one of the eigenvectors, you get the other by conjugating.

The important example:

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$ for real numbers a & b.

What is the corresponding transformation?



let's rewrite the matrix using r & θ.

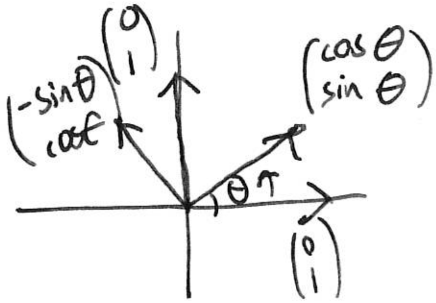
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = r \begin{pmatrix} a/r & -b/r \\ b/r & a/r \end{pmatrix}$$

$r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1} b/a.$

$$= r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

↖ what does this matrix do to a vector?



sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

it rotates a vector by θ counterclockwise!

so $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ does is: -rotates by angle θ

-then stretches by factor r.

what are the eigenvalues?

$$\det\begin{pmatrix} a-\lambda & -b \\ b & a-\lambda \end{pmatrix} = a^2 - 2a\lambda + \lambda^2 + b^2$$

$$= \lambda^2 + (-2a)\lambda + (a^2+b^2).$$

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = a \pm \sqrt{-b^2} = a \pm bi.$$

Slogan: if A is a matrix with eigenvalues $a \pm bi$,

it looks like a rotation by $\theta = \tan^{-1} b/a$, followed by

rescaling by factor $r = \sqrt{a^2+b^2}$.

(even if A is not the specific matrix above!)

(if you pick a suitable basis, it is rotation + scaling.)

otherwise this is a heuristic. ~~but~~

say you have a 2x2 matrix A, with complex eigenvalues

a-bi, a+bi.

$$\text{let } P = \left[\underbrace{\text{Real part}(\vec{v}_1)}_{\text{column 1}} \quad \underbrace{\text{Im. part}(\vec{v}_1)}_{\text{column 2.}} \right]$$

v_1=eigenvector for a-bi.

let $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (a&b from eigenvalues)

then $A = PCP^{-1}$.

like diagonalization!

A is similar to C, a matrix we understand.

eg..

$$A^{100}\vec{x} = PC^{100}P^{-1}\vec{x} = P\cdot \underset{\uparrow}{C^{100}}(P^{-1}\vec{x})$$

rotate by 100θ,
scale by r^{100}.

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\lambda = \frac{-1-\sqrt{3}i}{2}, \quad \frac{-1+\sqrt{3}i}{2}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \frac{-1-\sqrt{3}i}{2} \end{pmatrix}.$$

$\lambda = a - bi$ so $a = -\frac{1}{2}$ & $b = \frac{\sqrt{3}}{2}$

and $A = P C P^{-1}$

where $P = \begin{bmatrix} 1 & 0 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

Announcements

- Review today, 4-5, SEO 612
  tomorrow, 5-6 SEO 612.

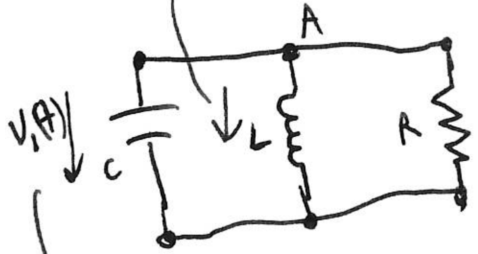
- Review sheet & practice test both updated.

- Exam W!

Today: coupled differential equations

Where these things come from. One example: RLC circuits.

$i_L(t)$ current across the inductor.



voltage across capacitor

(don't worry about this derivation if you haven't seen it before.)

Equations:

current across C.

(= i_R, by Ohm)

$$\frac{dv_1}{dt} = \frac{i_C}{C}, \quad \frac{di_L}{dt} = \frac{v_1}{L}, \quad i_C + i_L + \frac{v_1}{R} = 0$$

$\frac{d}{dt}$ "$q = Cv_1$"

Faraday's law

Kirkhoff

$$\frac{dv_1}{dt} = -\frac{i_L}{C} - \frac{v_1}{RC}$$

$$\frac{di_L}{dt} = \frac{v_1}{L}.$$

two functions $v_1(t)$, $i_L(t)$.

time derivative of each is just a linear combination of the two.

usetul to write this in matrix form.

$$\begin{pmatrix} \frac{dv_1}{dt} \\ \frac{di_L}{dt} \end{pmatrix} = \begin{pmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ i_L \end{pmatrix}.$$

"two coupled first order linear equations"

An easier one.

$x_1(t)$, $x_2(t)$ two functions, with

$$x_1'(t) = -x_1 + x_2 \qquad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$x_2'(t) = -2x_1 - 4x_2.$$

how to solve?

<u>make a substitution:</u>

$$z_1(t) = 2x_1 + x_2$$

$$z_2(t) = -x_1 - x_2.$$

these also satisfy some diff. eq.

$$z_1'(t) = \frac{d}{dt}(2x_1 + x_2) = 2x_1' + x_2' = 2(-x_1 + x_2) + (-2x_1 - 4x_2)$$

$$= -4x_1 - 2x_2 = -2z_1 \quad \text{(no } z_2\text{'s!)}$$

$$z_2'(t) = -x_1' - x_2' = -(-x_1 + x_2) - (-2x_1 - 4x_2)$$

$$= 3x_1 + 3x_2 = -3z_2. \quad \text{(no } z_1\text{'s!)}$$

our new eqns:

$$z_1' = -2z_1$$
$$z_2' = -3z_2 .$$

this we can solve!

$$z_1(t) = c_1 e^{-2t},$$
$$z_2(t) = c_2 e^{-3t}$$

now use this to get $x_1(t)$, $x_2(t)$:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{-3t} \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + c_2 e^{-3t} \\ -c_1 e^{-2t} - 2c_2 e^{-3t} \end{pmatrix}$$

why did this work?

our change of variables turned eqns involving two functions, into equations involving only one, which are easy.

"decoupling the system"

How did I know to use that change of variables?

→ diagonalization.

we had $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

let's diagonalize this matrix.

$$\det(A-\lambda I) = \begin{pmatrix} -1-\lambda & 1 \\ -2 & -4-\lambda \end{pmatrix} = (\lambda^2 + 5\lambda + 4) - (-2)$$
$$= \lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3).$$

$\lambda = -2$: $A - \lambda I = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \rightsquigarrow \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = -3$: $A - \lambda I = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \rightsquigarrow \vec{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

so $A = PDP^{-1}$, where

$$P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

↑ this looks familiar!

used $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

that's the substitution we used to decouple the system!

↓

so $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

This always works! Why?

make change of vars $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

given coupled equations: $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = PDP^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

set $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$$P^{-1}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = DP^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = D\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

because D is diagonal, this is <u>decoupled</u>.

easy to solve.

this is the equations $z_1' = \lambda_1 z_1$
$z_2' = \lambda_2 z_2$ where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Solution $\Rightarrow$ $z_1 = c_1 e^{\lambda_1 t}$, $z_2 = c_2 e^{\lambda_2 t}$.

then recover x_1 & x_2 from z_1, z_2:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

you end up with

$$x_1(t) = c_1 e^{\underbrace{\lambda_1 t + c_2 e^{\lambda_2}}_{\text{the } \lambda_1, \lambda_2 \text{ that show up are the eigenvalues of A!}} t}$$

What if we ended up with complex eigenvalues?

the way to handle this is a lot like what we did for diagonalization for complex numbers.

decoupled equations are

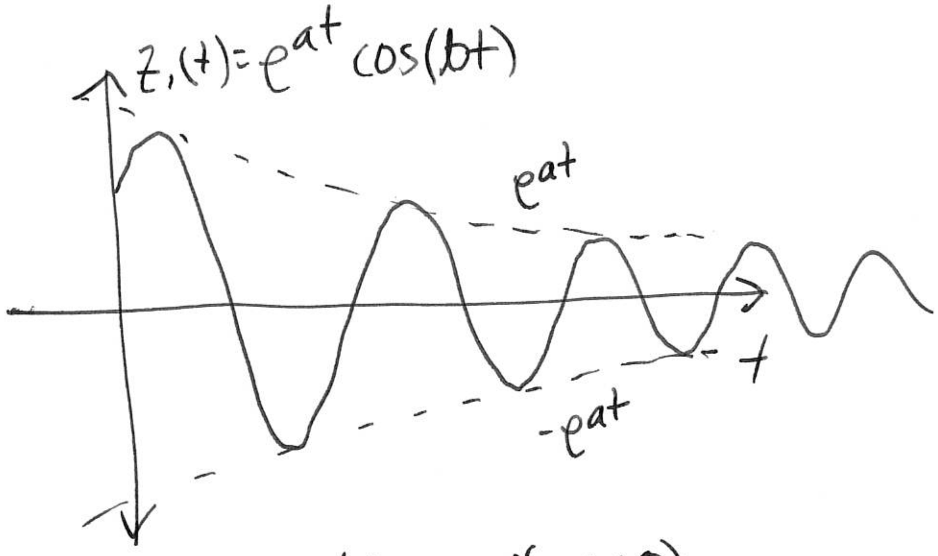
$$z_1' = (a+bi)z_1$$
$$z_2' = (a-bi)z_2$$

$$z_1 = e^{(a+bi)t} = e^{at+bit} \cancel{= \cos(at)}$$
$$= e^{at}(\cos(bt) + i\sin(bt)).$$

The real and imaginary parts are the solutions you want:

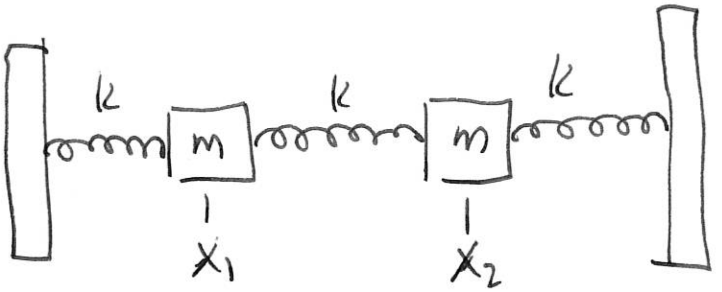
$$\underset{z_1}{\underset{\parallel}{e^{at}\cos(bt)}},\ \underset{z_2}{\underset{\parallel}{e^{at}\sin(bt)}}.$$

These look like:



(decays, if $a<0$).

One more example



masses & springs.

left spring   right spring

$$mx_1'' = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$$

$$mx_2'' = -k(x_2 - x_1) + k(-x_2) = kx_1 - 2x_2.$$

these give the equations.

let $y_1 = x_1$

define four new functions $y_2 = x_1'$

$y_3 = x_2$

$y_4 = x_2'$

$$y_1' = y_2$$

$$y_2' = -2ky_1 + ky_3$$

$$y_3' = y_4$$

$$y_4' = ky_1 - 2ky_3$$

in matrix form,

(with k=m=1)

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}.$$

you can solve this by diagonalizing to decouple the variables!

Announcements

- Next W's quiz: 5.5, 5.7, 6.1

  (no matrices in other coordinate systems)
  (5.4)

- Exams will be returned next week.

A little more diff eq

We were looking at diff eq's

$$\vec{x}' = A\vec{x}$$

e.g.

$$\overset{\vec{x}'}{\overset{\swarrow}{\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}}} = \begin{pmatrix} -x_1 + x_2 \\ -2x_1 - 4x_2 \end{pmatrix} = \overset{A}{\overset{\downarrow}{\begin{pmatrix} -1 & 1 \\ -2 & -4 \end{pmatrix}}} \overset{\vec{x}}{\overset{\downarrow}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}}$$

last time, we found:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + c_2 e^{-3t} \\ -c_1 e^{-2t} - 2c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}.$$

What are c_1 & c_2? they're parameters - any values gives a solution.

usually determined by initial conditions $\underset{\substack{\uparrow \\ \text{e.g. position of} \\ \text{mass at time 0.}}}{x_1(0)}, x_2(0)$.

suppose $x_1(0)=2$
$x_2(0)=-3$.

this determines c_1 & c_2.

plug in $t=0$ and see what it says.

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

find c_1 & c_2 using row reduction on augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -1 & -2 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right].$$

$c_1=1$

$c_2=1.$

so our solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t}.$$

Dealing with complex eigenvalues.

$\vec{x}' = A\vec{x}$, and you end up with $\lambda = a \pm bi$.

Pick one of the eigenvalues: $a + bi$.

One (complex) solution to the differential eqn

is $\vec{v}\, e^{(a+bi)t}$

$\uparrow$ complex eigenvector.

this makes $x_1(t)$ complex! we don't want that.

to get two real solutions, take the real part of $x_1(t)$, and the imaginary part.

e.g. $\begin{pmatrix}1+i\\2-i\end{pmatrix} e^{(3+2i)t} = \begin{pmatrix}1+i\\2-i\end{pmatrix}\left(e^{3t}(\cos 2t + i \sin 2t)\right).$

$$= \begin{pmatrix}(e^{3t}\cos 2t - e^{3t}\sin 2t) + (e^{3t}\cos 2t + e^{3t}\sin 2t)i \\ 2e^{3t}\cos 2t + e^{3t}\sin 2t) + (2e^{3t}\sin 2t - e^{3t}\cos 2t)i\end{pmatrix}$$

so our solutions are

real part:

$$y_1(t) = \begin{pmatrix} e^{3t}\cos 2t - e^{3t}\sin 2t \\ 2e^{3t}\cos 2t + e^{3t}\sin 2t \end{pmatrix} \begin{array}{l} \leftarrow x_1 \text{ for first sol} \\ \leftarrow x_2 \text{ for } \cancel{\text{second}}\ \overset{\text{first}}{} \text{ sol} \end{array}$$

imaginary:

$$y_2(t) = \begin{pmatrix} e^{3t}\cos 2t + e^{3t}\sin 2t \\ 2e^{3t}\sin 2t - e^{3t}\cos 2t \end{pmatrix}.$$

---

to find complex eigenvectors:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \det(A-\lambda I) \overset{\det}{=} \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$$= \lambda^2 + 1.$$

$$\lambda = i \text{ or } -i.$$

pick one. $(\lambda = +i)$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$ want $\vec{x}$ in nullspace.

$$\vec{x} = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Dot product

given two vectors of same size, $\vec{x}$ & $\vec{y}$,

$\vec{x}\cdot\vec{y}$ is a number.

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}4\\5\\6\end{pmatrix}=(1)(4)+(2)(5)+(3)(6)=32.$$

some facts:

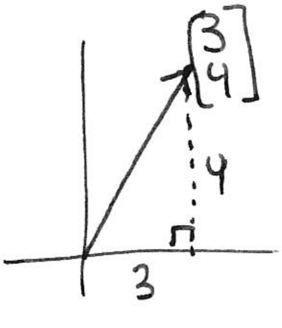
1. $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$

2. $\vec{u}\cdot\vec{u}$ is always $\geqslant 0$.

$$\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=\underbrace{a^2+b^2+c^2}_{\text{all } \geq 0.}$$

3. the length of $\vec{u}$ is $\sqrt{\vec{u}\cdot\vec{u}}$

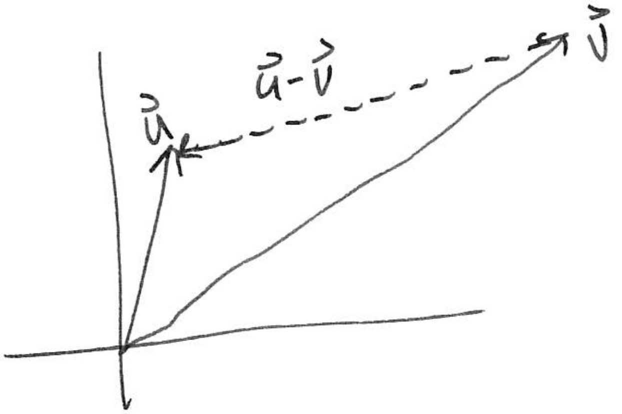
e.g.



length $=\sqrt{3^2+4^2}=\sqrt{25}=5$
(pythogorean thm)

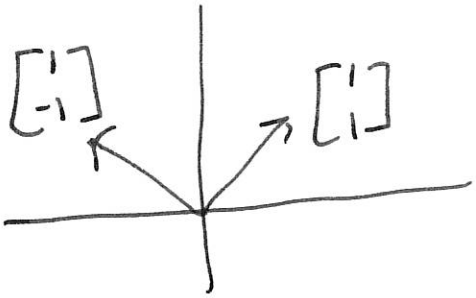
but $\begin{bmatrix}3\\4\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix}=3^2+4^2.$

4. it $\vec{u}$ & $\vec{v}$ are vectors, the distance between the ends is length$(\vec{u}-\vec{v}) = \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})}$



5. If $\vec{u}$ and $\vec{v}$ are perpendicular, then

$\vec{u}\cdot\vec{v}=0.$



$\begin{bmatrix}1\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\end{bmatrix}=(1)(1)+(-1)(1)=0.$

Orthogonal subspaces.

if $V \subset \mathbb{R}^n$ is a subspace, let $V^\perp$ be the subset of all vectors orthogonal to everything in V.

$V^\perp$ is a <u>subspace</u>, called orthogonal complement of V.

<u>ex</u> inside $\mathbb{R}^3$, $V =$ xy-plane = all vectors $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

$V^\perp$ is all vectors orthogonal to everything in x, y plane

- all vectors of the form $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$.

Reminder: where did these exponentials come from?

we did a change of variables, and ended up with the equations

$$y_1'(t) = -2y_1(t)$$

$$y_2'(t) = -3y_2(t)$$

so

$$y_1(t) = e^{-2t}.$$

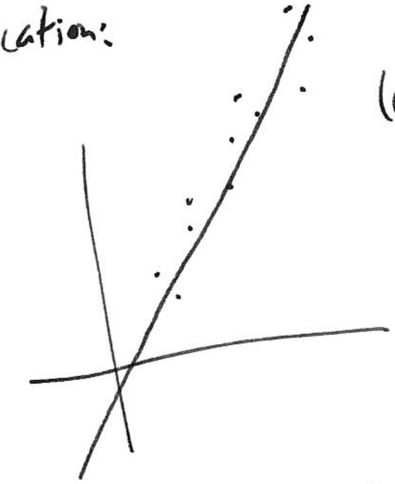
(because $y_1'(t) = (-2)e^{-2t} = -2e^{-2t}$.)

Announcements

- Exams will be returned W or F; sols posted already
- Quiz W: 5.5, 5.7, 6.1. HW sols all up as of this morning.

  (no matrices in other coords)
- OH Thursday cancelled; email to set up alternative.
- This week: Ch 6.

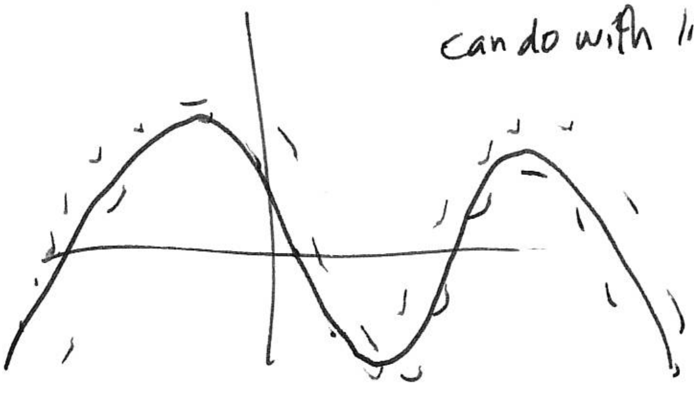
an application:



least-squares best fit line.

best fit sine wave?

can do with linear algebra!



Diff eqns: big picture

given diff eqn $\vec{x}' = A\vec{x}$

~~i.e.~~ eg. $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

what the solution looks like. find <u>two</u> (or 3, if 3×3...) basic solutions, and every other sol. is a linear combo of those two.

if the eigenvalues of A are real:

$$\underbrace{\vec{x} = \vec{v}_1 e^{\lambda_1 t}}_{\text{sol \#1}}, \qquad \underbrace{\vec{x} = \vec{v}_2 e^{\lambda_2 t}}_{\text{sol \#2.}}$$

(we found this by "decoupling" the system)

↓ shorthand for ↓

evector, evalue

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}. \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{7t}$$

general sol is ← every solution is of this form for some c_1 and c_2.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} e^{7t}$$

c_1 & c_2 are parameters. if $\vec{x}(0)$ is given, you can find c_1, c_2.

---

if eigenvals are complex, same idea, but different way to find basic sols:

$$\vec{x} = \vec{v}e^{\lambda t} \text{ is a solution, but complex.}$$

to get two basic real sols, multiply this out, take $\underbrace{\text{real part}}_{\#1 \text{ sol}}$, $\underbrace{\text{imaginary part}}_{\#2 \text{ sol.}}$.

every other solution is a combo of these two. (see F notes)

A note: if you use the other eigenvalue, get same two real solutions (with − on im. part).
so you only need to bother with one.

Last time: orthogonal complements.

$V \subset \mathbb{R}^3$ is subspace, then $V^\perp$ is all vectors orthogonal to V.

$V^\perp$ is a subspace too.

ex

$$V = \text{Span}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \text{all multiples of } \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

(a line)

$V^\perp$: plane of vectors perpendicular to $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

2-diml subspace

$$V = \text{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right), \text{ a plane.}$$

$V^\perp =$ ~~the~~ all vectors on line perp to this plane.

(1-diml subspace).

what's $\mathrm{Nul}\left(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}\right)$?

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 + 3x_3 \\ x_1 + \quad x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

it's all vectors with $1x_1 + 2x_2 + 3x_3 = 0$
$x_1 + x_3 = 0$

ie. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0.$

= all vectors orthogonal to both

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$= V^{\perp}$, where $V = \mathrm{span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$.

$\Rightarrow \boxed{\mathrm{Nul}(A) = \mathrm{Row}(A)^{\perp}}$ for any matrix A!

This lets you find a basis for orthogonal complement to the span of a set of vectors:

just use the vectors as rows of a matrix, and find basis for $\mathrm{Nul}(A)$.

e.g. basis for orthogonal complement of $U = \mathrm{span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)$

just compute basis for $\mathrm{Nul}\begin{pmatrix}1 & 2 & 3\\1 & 0 & 1\end{pmatrix}$.

§6.2.

A bunch of vectors is an <u>orthogonal set</u>

if any two are orthogonal.

(dot product of any two is 0).

e.g. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

e.g. $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -14 \\ -2 \\ 10 \end{pmatrix}.$

$$\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -14 \\ -2 \\ 10 \end{pmatrix} = -42 - 8 + 50 = 0.$$ (and all other combos)

orthogonal sets in $\mathbb{R}^3$ look like rotated version of normal coordinate axes.

these make good bases to work with; x-, y-, and z-directions are perpendicular.

What's the point?

Say you have a basis $\vec{b}_1, \vec{b}_2, \vec{b}_3$ fr $\mathbb{R}^3$,

and vectors are orthogonal set.

usually finding $[\vec{x}]_{\mathcal{B}}$ is a pain.

if $\vec{b}_i$'s are orthogonal, it's easy!

$$\vec{x} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3$$

where $c_1 = \dfrac{\vec{x} \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1}$

$\swarrow$ that's $[\vec{x}]_{\mathcal{B}}$!

$$c_2 = \frac{\vec{x} \cdot \vec{b}_2}{\vec{b}_2 \cdot \vec{b}_2}$$

$$c_3 = \frac{\vec{x} \cdot \vec{b}_3}{\vec{b}_3 \cdot \vec{b}_3}$$

Components of a vector.

given a vector $\vec{x}$ and second vector $\vec{u}$.

the component of $\vec{x}$ in the $\vec{u}$ direction is

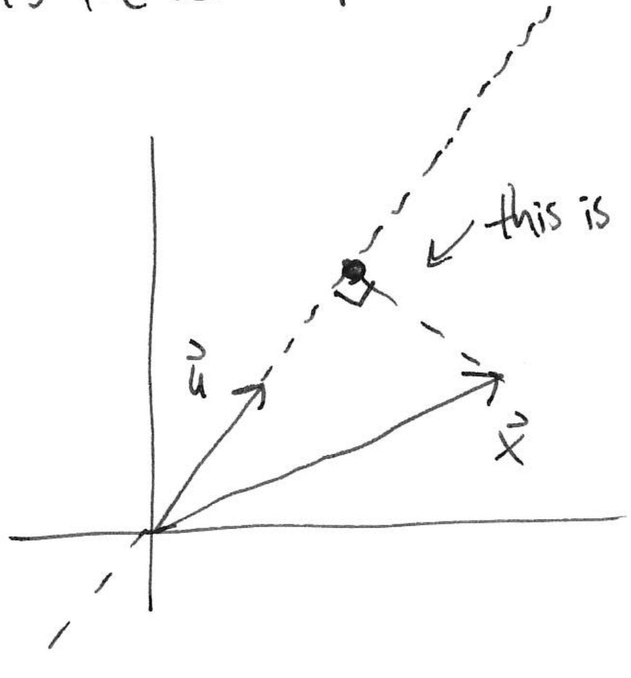
$$\overline{x} = \text{proj}_{\vec{u}}\vec{x} \quad \text{is} \quad \frac{\vec{x}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

↑

a scalar

$\text{proj}_{\vec{u}}\vec{x}$ is a vector in the direction of $\vec{u}$.

it's the vector parallel to $\vec{u}$ that's closest to $\vec{x}$.



this is $\text{proj}_{\vec{u}}\vec{x}$.

Announcements

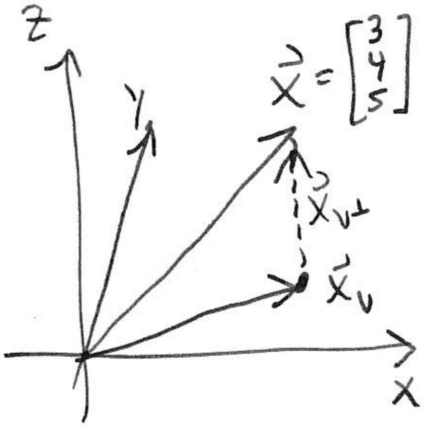
- Pick up your exam on the way out
- OH tomorrow cancelled; email to set a time. (Fri or Mon)
- Today: orthogonal projection.

Geometric observation:

$\vec{x}$ a vector in $\mathbb{R}^3$

$V =$ xy-plane, a subspace.

you can always write $\vec{x}$ as $\vec{x}_V + \vec{x}_{V^\perp}$, where $\vec{x}_V$ is in xy-plane, and $\vec{x}_{V^\perp}$ is perpendicular to V.



$$\vec{x}_V = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

$$\vec{x}_{V^\perp} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

the same thing can be done for any plane V, doesn't have to be xy-plane.

$$\vec{x} = \underset{\substack{\uparrow \\ \text{in } V}}{\vec{x}_V} + \underset{\substack{\uparrow \\ \text{in } V^\perp}}{\vec{x}_{V^\perp}}.$$

Finding $\vec{x}_V$, $\vec{x}_{V^\perp}$ is easy if V is xy-plane,

but there's a way to do it for any other plane.

If $W \subset \mathbb{R}^n$ is a subspace, and $\vec{w}_1, \dots, \vec{w}_p$ is an orthogonal basis for W, then any vector $\vec{x}$ can be written as $\vec{x}_W + \vec{x}_{W^\perp}$, where $\vec{x}_W$ in W, $\vec{x}_{W^\perp}$ perp to W.

$$\vec{x}_W = \underbrace{\frac{\vec{x}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}}_{\text{scalar}}\vec{w}_1 + \underbrace{\frac{\vec{x}\cdot\vec{w}_2}{\vec{w}_2\cdot\vec{w}_2}}_{\text{scalar}}\vec{w}_2 + \dots + \underbrace{\frac{\vec{x}\cdot\vec{w}_p}{\vec{w}_p\cdot\vec{w}_p}}_{\text{scalar}}\vec{w}_p.$$

then $\vec{x}_{W^\perp} = \vec{x} - \vec{x}_W$.

---

x_W is definitely in W, since it's a linear combo of basis vectors.

Example

$W = xy$-plane.

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

Formula says:

$$\vec{x}_W = \frac{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{3}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{4}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \quad \checkmark$$

$$\vec{x}_{W^\perp} = \vec{x} - \vec{x}_W = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}.$$

Notes:

1. $\vec{x}_W$ is also called $\text{proj}_W \vec{x}$.

2. $\vec{x}_W$ is the vector in W that's as close as possible to $\vec{x}$.

   (distance as small as possible)

3. - the formula required an orthogonal basis!
   - our methods for finding bases don't give orthogonal bases, most of the time.
   - Fri: Gram-Schmidt ~~elimination:~~ orthonormalization: given a basis, convert to an orthogonal basis. this can be used in the formula.

Announcements

- next quiz will be a take-home; stay tuned.
- we'll have a full lecture on W to finish 6.5 & 6.6, then do some general review + applications if there's time.
- please do your course evals! you should have gotten an email already.

A set of vectors $\vec{v}_1, \dots, \vec{v}_r$ is called <u>orthonormal</u> if the vectors are all orthogonal, and length 1.

→ it's easy to turn an orthogonal set into an orthonormal set: just divide each one by its length.

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

orthogonal. but $\|\vec{x}_1\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$

$\|\vec{x}_2\| = 1.$

$\vec{v}_1 = \vec{x}_1 / \|\vec{x}_1\| = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$ and $\vec{v}_2 = \vec{x}_2 / \|\vec{x}_2\| = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is orthonormal.

if U is a matrix $(m \times n)$, with orthonormal cols,

then $U^T U = n \times n$ identity matrix.

$$U = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix}$$

$$U^T U = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}}_{U^T} \underbrace{\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{pmatrix}}_{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

<u>Fact</u>: If $\vec{u}_1, \dots, \vec{u}_n$ is an orthonormal basis for a subspace $W \subseteq \mathbb{R}^m$ (eg $\vec{u}_1, \vec{u}_2$, basis for plane in $\mathbb{R}^3$)

another formula for projection:

$$\text{proj}_W \vec{x} = (U U^T)\vec{x},$$

where U is a matrix with basis vector as columns.

Note: if U is $m \times n$, then $U^T U$ is $n \times n$ identity
but $U U^T$ is $m \times m$, not identity!

don't mix them up!

if U is square $n \times n$, $U U^T$ is identity too.

Gram-Schmidt orthonormalization.

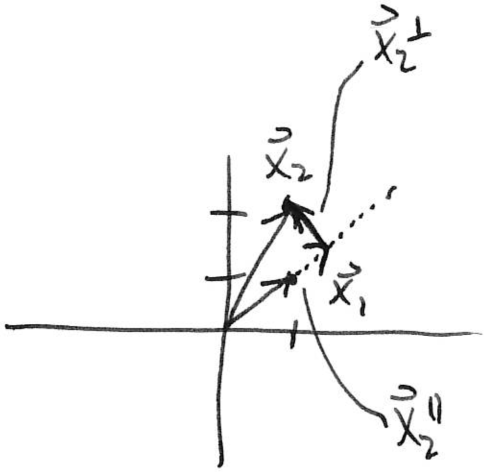
input: basis for a subspace $W \subset \mathbb{R}^n$

output: orthonormal basis for same subspace, which you can use in our formulas.

2D example.

$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



to start, take original vector, and don't change it.

$\vec{v}_1 = \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ write $\vec{x}_2$ as $\underbrace{\vec{x}_2^{\parallel}}_{\text{parallel to } \vec{x}_1} + \underbrace{\vec{x}_2^{\perp}}_{\text{perp to } \vec{x}_1\text{; use as 2nd basis vector.}}$

$$\vec{V}_2 = \vec{x}_2^{\perp} = \vec{x}_2 - \vec{x}_2^{\parallel}$$

$$= \vec{x}_2 - \underbrace{\frac{\vec{x}_2 \cdot \vec{x}_1}{\vec{x}_1 \cdot \vec{x}_1} \vec{x}_1}_{\text{proj}_{\vec{x}_1} \vec{x}_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$$

$\vec{u}_1 = \vec{V}_1 / \|\vec{v}_1\| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

to get an orthonormal basis: $\vec{u}_2 = \vec{v}_2 / \|\vec{v}_2\| = \frac{2}{\sqrt{2}} \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}$.

In general, say you're handed $\vec{x}_1, \dots, \vec{x}_n$ basis for $W \subset \mathbb{R}^m$.

(eg $m=3$, $n=2$: a plane in $\mathbb{R}^3$)

---

take

<u>Step 1</u>: get orthogonal basis.

take $\vec{v}_1 = \vec{x}_1$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

($\vec{v}_1$ & $\vec{v}_2$ are orthogonal)

$$\vec{v}_3 = \vec{x}_3 \underbrace{- \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2}$$

this is the component of $\vec{x}_3$ parallel to $\vec{v}_1$ & $\vec{v}_2$; after subtracting, what's left is orthogonal to both.

...

(good picture on page 357)

<u>Step 2</u>:

the $\vec{v}_i$'s are an orthogonal basis, but not orthonormal.

an orthonormal basis is given by

$$\vec{u}_1 = \vec{v}_1 / \|\vec{v}_1\|$$

$$\vec{u}_2 = \vec{v}_2 / \|\vec{v}_2\|$$

...

Example:

$$\underset{\substack{\uparrow\\ \vec{x}_1}}{\begin{pmatrix}1\\0\\1\\2\end{pmatrix}},\quad \underset{\substack{\uparrow\\ \vec{x}_2}}{\begin{pmatrix}3\\1\\5\\5\end{pmatrix}},\quad \underset{\substack{\uparrow\\ \vec{x}_3}}{\begin{pmatrix}1\\-2\\3\\4\end{pmatrix}}$$

basis for a
3-diml subspace.
of $\mathbb{R}^4$

$$\vec{v}_1 = \vec{x}_1 = \begin{pmatrix}1\\0\\1\\2\end{pmatrix}$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \begin{pmatrix}3\\1\\5\\5\end{pmatrix} - \frac{18}{6}\begin{pmatrix}1\\0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\2\\-1\end{pmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \begin{pmatrix}1\\-2\\3\\4\end{pmatrix} - \frac{12}{6}\begin{pmatrix}1\\0\\1\\2\end{pmatrix} - \frac{0}{6}\begin{pmatrix}0\\1\\2\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\-2\\1\\0\end{pmatrix}.$$

to get an orthonormal basis:

$$\vec{u}_1 = \vec{v}_1/\|\vec{v}_1\| = \begin{pmatrix}1/\sqrt{6}\\0\\1/\sqrt{6}\\2/\sqrt{6}\end{pmatrix},\quad \vec{u}_2 = \vec{v}_2/\|\vec{v}_2\| = \begin{pmatrix}0\\1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix},\quad \vec{u}_3 = \vec{v}_3/\|\vec{v}_3\| = \begin{pmatrix}-1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\\0\end{pmatrix}$$

QR factorization

if A is $m \times n$, lin indep cols. then can write

$A = QR$ where:

- Q $m \times n$, orthonormal cols.
- R $n \times n$, upper triangular.

---

cols of Q = result of Gram-Schmidt, on cols of A.

$R = Q^T A$. (just multiply it out.)

---

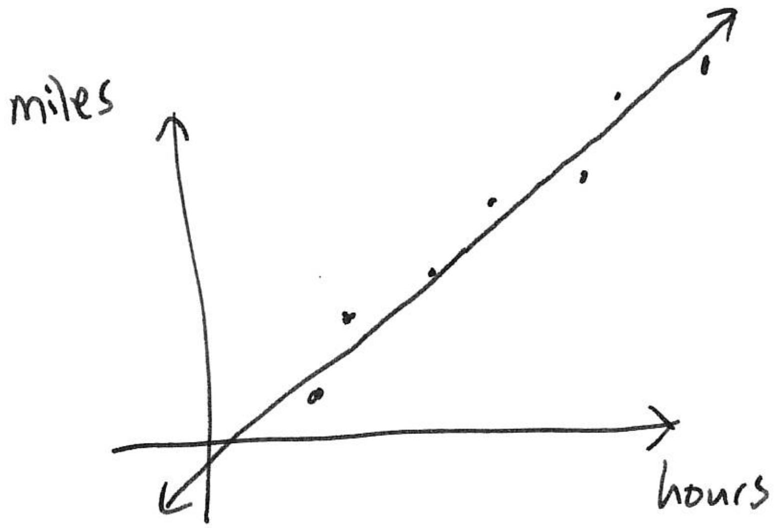
this is sort of similar to LU decomposition, and simplifies some calculations. Whereas LU decomposition "remembers" how to do row reduction, QR factorization "remembers" how to do Gram-Schmidt.

Announcements

- Take-home quiz, due W. (posted on site, or take a copy today.)
- Back to normal next week.
- Today & Wed: 6.5 & 6.6

Best-fit lines

Example you're on a road trip, check odometer every hour



your measured position is roughly a linear function of time:

$$\text{miles} = r \cdot t + b \quad \text{for some } r \leftarrow \text{average speed}$$
$$b \leftarrow \text{starting position}$$

you can try to guess r&b to fit a line.

1) what's my average speed?
   about r

2) where will I be at $t=10$?
   plug in $t=10$: about $10r+b$.

Today: - how to find a formula for the line.
- or for parabolas etc!

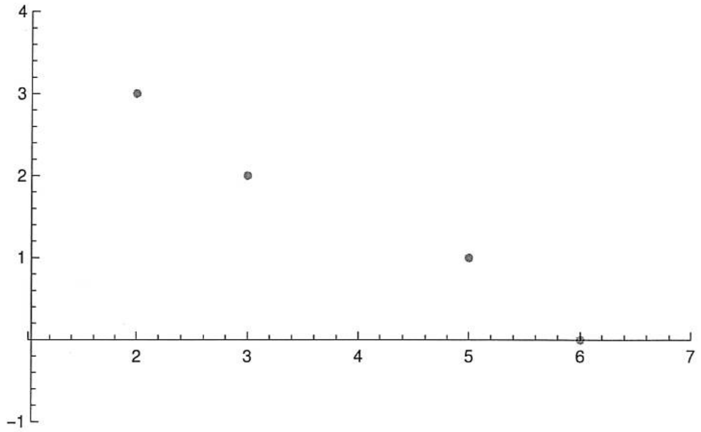
say we want a line to fit the 4 points (2,3), (3,2), (5,1), (6,0)

```
In[6]:= points = {{2, 3}, {3, 2}, {5, 1}, {6, 0}}

Out[6]= {{2, 3}, {3, 2}, {5, 1}, {6, 0}}

In[7]:= mypoints = ListPlot[points, PlotRange → {{1, 7}, {-1, 4}}]
        myline = Plot[-7 / 10 x + 43 / 10, {x, 1, 7}];
```

Out[7]=

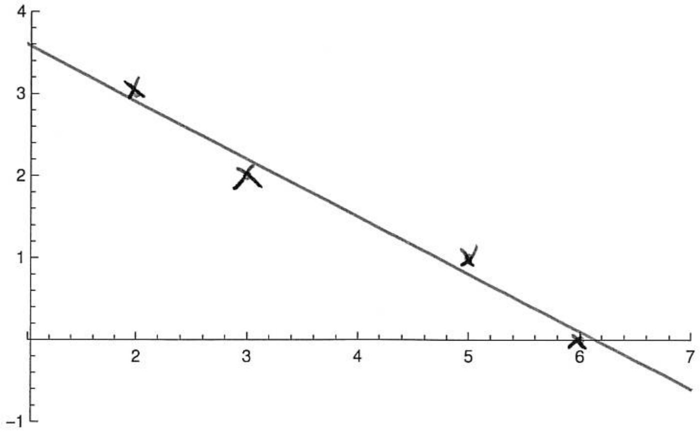


```
In[9]:= RowReduce[{{2, 1, 29 / 10}, {3, 1, 22 / 10}, {5, 1, 8 / 10}, {6, 1, 1 / 10}}]

               7            43
Out[9]= {{1, 0, - ──}, {0, 1, ──}, {0, 0, 0}, {0, 0, 0}}
               10           10

In[10]:= Show[mypoints, myline]
```

Out[10]=



```
In[11]:= quadpts = {{0, 2}, {1, 1}, {2, 1}, {3, 1}, {4, 3}}

Out[11]= {{0, 2}, {1, 1}, {2, 1}, {3, 1}, {4, 3}}

In[12]:= matr = {{0, 0, 1}, {1, 1, 1}, {4, 2, 1}, {9, 3, 1}, {16, 4, 1}}

Out[12]= {{0, 0, 1}, {1, 1, 1}, {4, 2, 1}, {9, 3, 1}, {16, 4, 1}}

In[13]:= Inverse[Transpose[matr].matr].{62, 18, 8}

          3    53  72
Out[13]= {─, - ──, ──}
          7    35  35
```

let's say our line is $y=mx+b$.

how to find m&b?

to go through (2,3) we want $3=2m+b$.

(3,2) $2=3m+b$

(5,1) $1=5m+b$

(6,0) $0=6m+b$.

in matrix form,

$$\begin{array}{ccc} A & \vec{x} & = \vec{b}. \\ \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix} & \begin{pmatrix} m \\ b \end{pmatrix} & = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \end{array}$$

this has no solutions! (too many eqns, too few variables)

$A\vec{x}=\vec{b}$, no solutions.

in linalg terms, the problem is that $\vec{b}$ is not in column space of A, i.e. $\vec{b}$ not a combo of columns of A.

Instead: solve $A\vec{x}=\hat{b}$, where $\hat{b}$ is the closest thing to $\vec{b}$ that <u>is</u> in the column space, so there is a sol.

(we know how to find $\hat{b}$!)

what's the closest thing to $\vec{b}$ in Col A?

$$A=\begin{pmatrix}2&1\\3&1\\5&1\\6&1\end{pmatrix} \quad \vec{b}=\begin{pmatrix}3\\2\\1\\0\end{pmatrix}.$$

it's $\text{proj}_{\text{Col } A}\vec{b}$. to find it: find an orthonormal basis for Col A!

then use formula for projection.

original basis:

$$\vec{x}_1=\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \vec{x}_2=\cancel{\begin{pmatrix}3\\2\\1\\0\end{pmatrix}}. \begin{pmatrix}2\\3\\5\\6\end{pmatrix}$$

$$\vec{v}_1=\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$\vec{v}_2=\vec{x}_2-\frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1=\cancel{\begin{pmatrix}3\\2\\1\\0\end{pmatrix}-\frac{6}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}=$$

$$=\begin{pmatrix}2\\3\\5\\6\end{pmatrix}-\frac{16}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}.$$

this is an orthogonal basis!

$$\operatorname{proj}_{\text{Col }A}\underset{\overset{\uparrow}{\vec{b}}}{\begin{pmatrix}3\\2\\1\\0\end{pmatrix}}$$

$$=\frac{\begin{pmatrix}3\\2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}+\frac{\begin{pmatrix}3\\2\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}}{\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}}\begin{pmatrix}-2\\-1\\1\\2\end{pmatrix}=\begin{pmatrix}29/10\\22/10\\8/10\\1/10\end{pmatrix}.$$

$$\uparrow$$

$$\frac{\vec{b}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1+\frac{\vec{b}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

this is in Col A.
and pretty close to $\vec{b}$!

now want to solve $A\vec{x}=\hat{b}$,
where $\hat{b}$ is the new thing.

$$\left(\begin{array}{cc|c}2&1&29/10\\3&1&22/10\\5&1&8/10\\6&1&1/10\end{array}\right)\xrightarrow[\text{reduce}]{\text{row}}\left(\begin{array}{cc|c}1&0&-7/10\\0&1&43/10\\0&0&0\\0&0&0\end{array}\right)$$

it's consistent

$\vec{x}=\begin{pmatrix}m\\b\end{pmatrix}$, so $m=-7/10$

$b=43/10$

This is the general set-up of a least squares problem. you'd like to solve $A\vec{x}=\vec{b}$ but there are no solutions: $\vec{b}$ not in Col A.

Instead, solve $A\vec{x}=\hat{b}$, where $\hat{b}=\text{proj}_{\text{Col } A}\vec{b}$.
the solution $\vec{x}$ (usually written $\hat{x}$) is the least-squares solution to $A\vec{x}=\vec{b}$.

it's the $\hat{x}$ such that $\underset{\substack{\uparrow \\ \text{distance between } A\hat{x} \text{ and } \vec{b}.}}{\|A\hat{x}-\vec{b}\|} < \|A\vec{x}-\vec{b}\|$ for any other $\vec{x}$.

i.e. any other $\vec{x}$ makes the error larger.

There's a simpler formula:

if you want to solve $A\vec{x}=\vec{b}$, but you can't, instead solve $A^TA\vec{x}=A^T\vec{b}$: this gives least squares sol.

in least squares, A has small # of cols, large # of rows e.g. 2×100. but A^TA is then 2×2 square matrix.

e.g.

$$\underset{A}{\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}}$$

$$\Downarrow$$

$$\underset{A^T}{\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}} \underset{A}{\begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \underset{A^T}{\begin{pmatrix} 2 & 3 & 5 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix}} \underset{\vec{b}}{\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}}$$

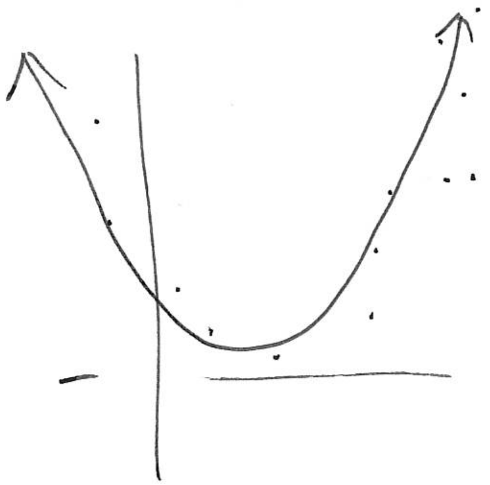
$$\begin{pmatrix} 74 & 16 \\ 16 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 17 \\ 6 \end{pmatrix}.$$

this has solutions!

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -7/10 \\ 43/10 \end{pmatrix}$$

(matches our original answer)

you can~~t~~ use this to fit other kinds of functions too!



eg fit a parabola

$at^2 + bt + c$

more generally, to fit

$a_1 t^3 + a_2 \sin t + a_3 e^{7t}$,

you can use this strategy.

the first step: write down the formulas you want to be satisfied (e.g. what condition on a, b, c makes it go through points exactly).

usually no solution, but find least squares sol.

Announcements

- Quiz due today at 5. (or in class, if you're ready)

  (if jdl@uic.edu bounces, try john.lesieutre@gmail.com)

- plan for the final week:

  M 11/30: 7.1 Diagonalization of symmetric matrices (on exam)
  W 12/2: 7.4 SVD (not on); Quiz, 6.5&6.6
  F 12/4: Review (extra office hrs TBA)

  practice exam will be up by Monday, or email me!

  etc.

- Last quiz will be returned M. (sorry!)

Summary of Monday

- Given $A\vec{x}=\vec{b}$ with no solutions, you can instead

  solve $A\hat{x}=\hat{b}$, where $\hat{b}=\text{proj}_{\text{Col}\,A}\vec{b}$.

  $\hat{b}$ should be close to $\vec{b}$, but there will be sols.

- A quicker way to do this: solve:

$$\underbrace{A^T A}_{\text{square matrix, usually invertible.}}\vec{x}=A^T\vec{b}$$

this is called the "normal equations" for $A\vec{x}=\vec{b}$

Main application: best-fit lines, best-fit parabolas, etc.

- you have a bunch of points $(x_1,y_1),(x_2,y_2),\dots$
- want to fit a parabola (etc) closest to the points:

1. write down the linear equations the coefficients of the parabola would satisfy if it went through all pts exactly.

2. this will be a linear system $A\vec{x}=\vec{b}$, but probably no sols. take least squares sol instead.

let's try to fit a parabola at^2+bt+c

through the points $(0,2),(1,1),(2,1),(3,1),(4,3)$

---

goal: solve for a, b, c.

to go through $(0,2)$ means if we plug in $t=0$, output should be 2.

$$0a+0b+c=2.$$

" $(1,1)$ " plug in $t=1$, output should be 1.

$$a+b+c=1.$$

to go through $(2,1)$:

$$4a+2b+c=1.$$

$(3,1)$: $9a+3b+c=1$

$(4,3)$: $16a+4b+c=3.$

in matrix form:

$$\underset{A}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}}$$

$$A \quad \vec{x} = \vec{b}$$

no sols: use the normal equations:

$$A^T A\vec{x} = A^T\vec{b}$$

$$\begin{pmatrix} 354 & 100 & 30 \\ 100 & 30 & 10 \\ 30 & 10 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 62 \\ 18 \\ 8 \end{pmatrix}.$$

solution is: $\begin{pmatrix} 3/7 \\ -53/35 \\ 72/35 \end{pmatrix}.$

parabola is $y = \frac{3}{7}x^2 - \frac{53}{35}x + \frac{72}{35}$

$(x = t)$

```
quadpts = {{0, 2}, {1, 1}, {2, 1}, {3, 1}, {4, 3}}
```

```
{{0, 2}, {1, 1}, {2, 1}, {3, 1}, {4, 3}}
```

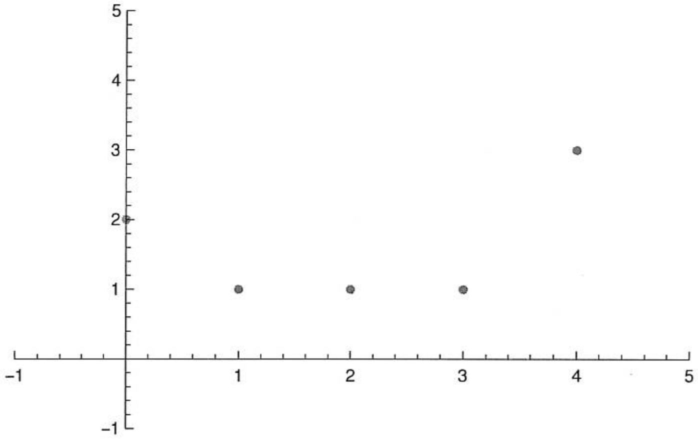
```
matr = {{0, 0, 1}, {1, 1, 1}, {4, 2, 1}, {9, 3, 1}, {16, 4, 1}}
Transpose[matr].{2, 1, 1, 1, 3}
Inverse[Transpose[matr].matr].{62, 18, 8}
```

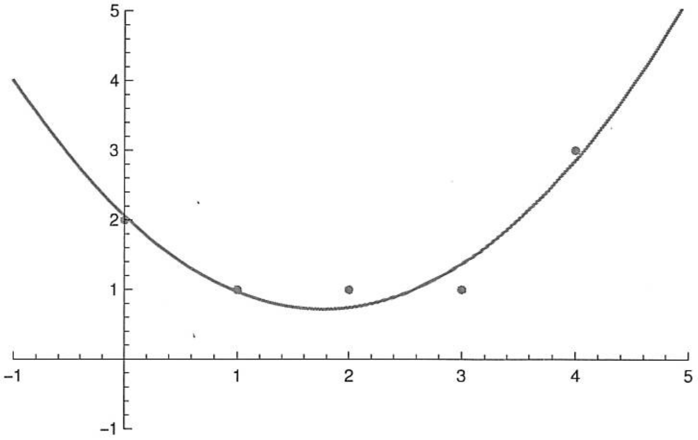
```
{{0, 0, 1}, {1, 1, 1}, {4, 2, 1}, {9, 3, 1}, {16, 4, 1}}
```

```
{62, 18, 8}
```

$$\left\{\frac{3}{7}, -\frac{53}{35}, \frac{72}{35}\right\}$$

```
mypoints = ListPlot[quadpts, PlotRange → {{-1, 5}, {-1, 5}}]
myline = Plot[3 / 7 x^2 - 53 / 35 x + 72 / 35, {x, -1, 5}];
Show[mypoints, myline]
```





when you're doing a best-fit curve, you end up with an equation $A\vec{x}=\vec{b}$.

A is the "design matrix"

((comes from equations your coefficients have to satisfy)

row for every point you're trying to interpolate (many)

columns for each coef in equation for curve. (few)

$\vec{x}$ is called "parameter vector"

this is what you're trying to find; entries are the unknown coefficients.

$\vec{b}$ is the "observation vector"

---

in the book, different notation: $X\vec{\beta}=\vec{y}$

$$\begin{tikzcd}[column sep=small] X & \vec{\beta} & = & \vec{y} \\ A \arrow[u] & \vec{x} \arrow[u] & & \vec{b} \arrow[u] \end{tikzcd}$$

the vector $\vec{\epsilon}=\vec{b}-A\vec{x}$ is called vector of residuals.

it's the error: how far is our vector from the points

we choose $\vec{x}$ to make this as small as possible.

it $A=QR$ is QR factorization, it's quick to compute least squares sol to $A\vec{x}=\vec{b}$.

it's given by $\underline{\hat{x}=R^{-1}Q^T\vec{b}}$.

| computationally this is useful.

Fun facts about A^TA

if A is any matrix, $m\times n$ (maybe not square)

- A^TA is square, $n\times n$. (usually not identity)
- if A has orthonormal cols, it's the identity matrix.
- A^TA is always symmetric!
  - $(A^TA)^T=A^T(A^T)^T=A^TA$
- if A has lin indep cols, then A^TA is invertible (which means $(A^TA)\vec{x}=A^T\vec{b}$ has only one sol.)

---

where does the formula come from?

solve: $A^TA\hat{x}=A^T\vec{b}$ $\quad A=QR$

$$R^T\underbrace{Q^TQ}_{I}R\vec{x}=R^TQ^T\vec{b} \leftarrow A^T=R^TQ^T$$

$$\cancel{R^T}R\vec{x}=\cancel{R^T}Q^T\vec{b}$$

$$R\vec{x}=Q^T\vec{b} \Rightarrow \hat{x}=R^{-1}Q^T\vec{b}$$

You can do best fit of functions with multiple inputs!

suppose you want to fit a plane $z = ax + by + c$

through the points:

$(1,1,5)$
$(1,-1,-2)$
$(0,0,1)$
$(-3,1,-4)$
$(1,3,10)$

want to solve for (a,b,c).

each point gives you an equation relating a,b,c.

$a+b+c=5$
$a-b+c=-2$
...

matrix form: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ -3 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \\ -4 \\ 10 \end{pmatrix}$.

least squares sols: $a = 25/12 \approx 2$
$b = 125/44 \approx 3$
$c = -3/11 \approx 0$.

$$z = \frac{25}{12}x + \frac{125}{44}y - \frac{3}{11}$$

$\approx 2x + 3y$, seems about right.

Announcements

- Today: 7.1

  Wed: 7.4, for fun + Quiz (6.5 + 6.6)

  Fri: review, OH all day

  Next Mon: exam! 1-3, room TBA

- Everything is posted, but quizzes delayed again. Very sorry! We will get everything returned by the end of the week.

- Please fill out your course evals!

7.1 symmetric matrix.

A square matrix A $(n\times n)$ is symmetric
if $A^T = A$ (flip it over, and it doesn't change)

$$A = \begin{pmatrix} 2 & 3 & -5 \\ 3 & 7 & 2 \\ -5 & 2 & 1 \end{pmatrix}$$

these show up a lot: - A^TA is always symmetric.
- moment of inertia tensor
- quadratic forms (7.2)

<u>Fact:</u> if A is symmetric matrix, and $\vec{v}, \vec{w}$ are eigenvectors with different eigenvals, then they're orthogonal to each other.

<u>Why?</u> say $A\vec{v} = \lambda\vec{v}$, $A\vec{w} = \mu\vec{w}$, and $\lambda \neq \mu$.

$$\vec{v}\cdot\vec{w} = \vec{v}^T\vec{w} = \left(\tfrac{1}{\lambda}A\vec{v}\right)^T\vec{w} = \tfrac{1}{\lambda}\vec{v}^TA^T\vec{w}$$

$$= \tfrac{1}{\lambda}\vec{v}^TA\vec{w} = \frac{\mu}{\lambda}\vec{v}^T\vec{w} = \frac{\mu}{\lambda}(\vec{v}\cdot\vec{w})$$

this is only possible if $\vec{v}\cdot\vec{w} = 0$!

Example

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 9 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 4 \\ 4 & 9-\lambda \end{pmatrix} = \lambda^2 - 12\lambda + 11$$
$$= (\lambda - 11)(\lambda - 1).$$

$\lambda = 11.$

$$A - 11I = \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \rightsquigarrow \vec{v}_1 = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\lambda = 1.$

$$A - 1I = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} \rightsquigarrow \vec{v}_2 = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2 - 2 = 0. \text{ as expected}$$

---

it you normalize the eigenvectors, you get $A = PDP^{-1}$ where P has orthonormal columns.

$$A = PDP^{-1}, \quad P = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix} \quad D = \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$$

This is a surprise!

usually you can't tell if a matrix is diagonalizable just by looking.

Spectral theorem

if A is an $n \times n$ symmetric matrix:

1) A has n real eigenvalues (no complex ones)
   (might be repeated, though)

2) You can always find a basis of eigenvectors
   (if eigenval shows up twice, you can find two lin indep eigenvectors): $\mathrm{Nul}\,(A-\lambda I)$ 2-dim'l.

3) The eigenspaces for different eigenvals are orthogonal to each other.

4) A is "orthogonally diagonalizable":
   $A = PDP^{-1}$, where P has orthonormal cols.

An application from 7.2. (not on exam)

say you want to graph

$$3x^2+8\underset{\uparrow}{\underline{\underline{xy}}}+9y^2=11$$

there's an xy term, so you can't write it as

$$(x-a)^2+(y-b)^2=c^2$$ (which you know how to graph)

what will happen: you get an ellipse not lined up with x&y axes. need to diagonalize to find the answer!

Idea: similar to decoupling a diff eq. first, rewrite the equation:

$$(x\ y)\begin{pmatrix}3 & 4\\ 4 & 9\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=11.$$

$$=(x\ y)\begin{pmatrix}3x+4y\\ 4x+9y\end{pmatrix}\overset{?}{=}3x^2+4xy+4xy+9y^2$$

$$=3x^2+8xy+9y^2$$

$$= (x\ y)\ PDP^{-1}\begin{pmatrix}x\\y\end{pmatrix} = 11$$

$$= (x\ y)\begin{pmatrix}1/\sqrt{5} & +2/\sqrt{5}\\ 2/\sqrt{5} & -1/\sqrt{5}\end{pmatrix}\begin{pmatrix}11 & 0\\ 0 & 1\end{pmatrix}\underbrace{\begin{pmatrix}1/\sqrt{5} & 2/\sqrt{5}\\ 2/\sqrt{5} & -1/\sqrt{5}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}}_{P^{-1}\vec{x}.}.$$

change coordinates:

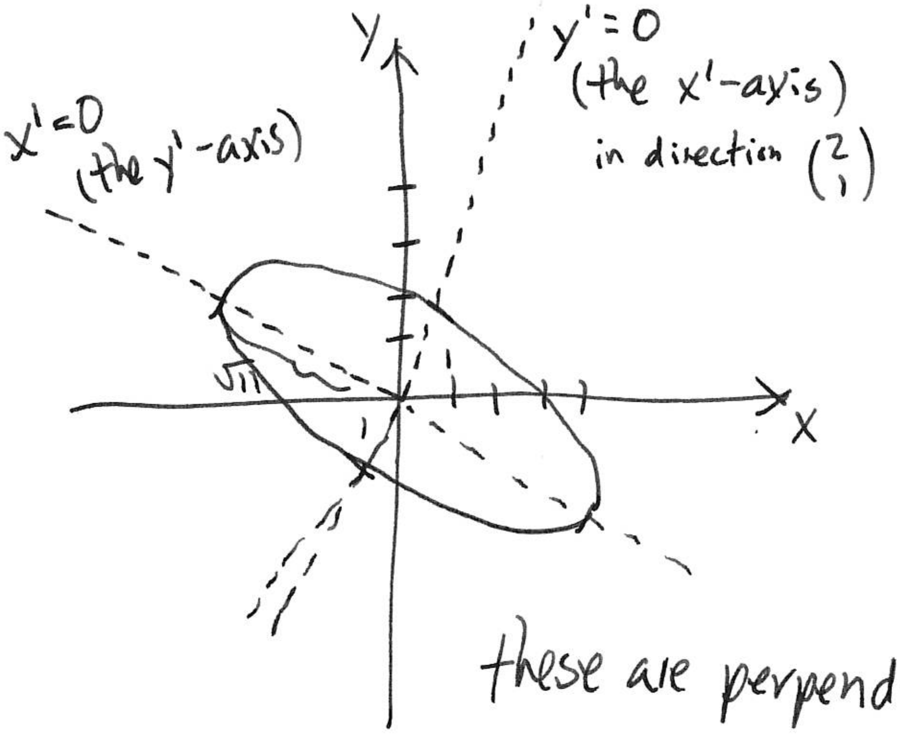
let $x' = 1/\sqrt{5}\,x + 2/\sqrt{5}\,y$

$y' = 2/\sqrt{5}\,x - 1/\sqrt{5}\,y.$ (like decoupling)

new eqn is

$$(x'\ y')\begin{pmatrix}11 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix} = 11$$

$$11(x')^2 + (y')^2 = 11$$



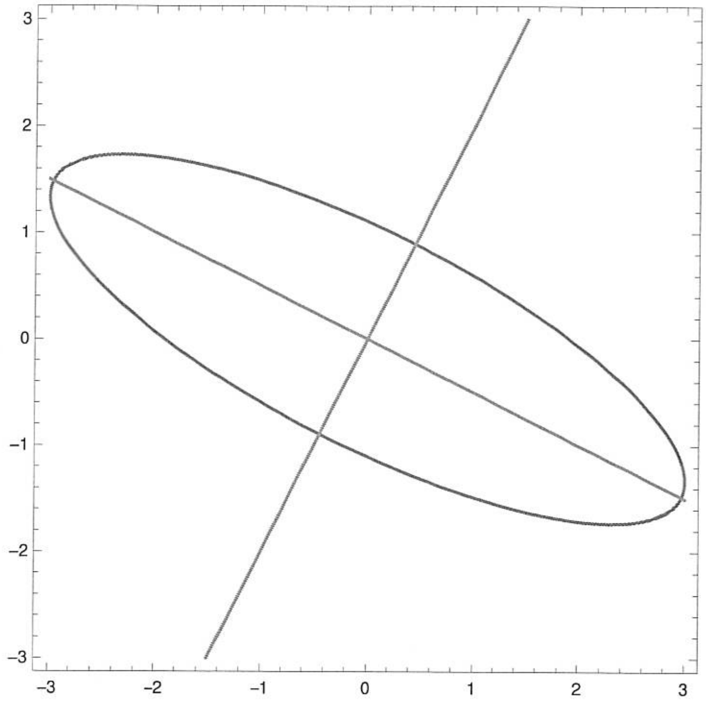
these are perpendicular!

graph your ellipse using normal method, but in x', y' coords.

Computer plot:

```
In[7]:= ContourPlot[{3 x^2 + 8 x y + 9 y^2 == 11, y == 2 x, y == -1 / 2 x}, {x, -3, 3}, {y, -3, 3}]
```

Out[7]=



Shortcut for eigenvectors of a 2x2 matrix.

When you write down $A-\lambda I$, row 2 is a mult of row 1.

e.g $A-\lambda I = \begin{pmatrix} 7 & 3 \\ 21 & 9 \end{pmatrix}$

pick a nonzero row, switch the entries, add a - sign.

eg. $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ is a λ-eigenvalue.

since we switched entries

then $\begin{pmatrix} 7 & 3 \\ 21 & 9 \end{pmatrix}\begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- This is only for 2x2!
- Works because $A-\lambda I$ has a nullspace
- OK for complex eva's too