9.2 Eigenanalysis II

Discrete Dynamical Systems

The matrix equation

$$\vec{\mathbf{y}} = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix}\vec{\mathbf{x}} \tag{1}$$

predicts the state $\vec{\mathbf{y}}$ of a system initially in state $\vec{\mathbf{x}}$ after some fixed elapsed time. The 3×3 matrix A in (1) represents the **dynamics** which changes the state $\vec{\mathbf{x}}$ into state $\vec{\mathbf{y}}$. An equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ like equation (1) is called a **discrete dynamical system** and A is called a **transition matrix**, provided A has nonnegative entries and column sums equal to one (see **Stochastic Matrices** below).

The eigenpairs of A in (1) are shown in *details* page 658 to be $(1, \vec{\mathbf{v}}_1)$, $(1/2, \vec{\mathbf{v}}_2)$, $(1/5, \vec{\mathbf{v}}_3)$ where the eigenvectors are given by

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 5/4 \\ 13/12 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix}. \tag{2}$$

Market Shares

A typical application of discrete dynamical systems is telephone long distance company market shares x_1, x_2, x_3, which are fractions of the total market for long distance service. If three companies provide all the services, then their market fractions add to one: $x_1 + x_2 + x_3 = 1$. The equation $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ gives the market shares of the three companies after a fixed time period, say one year. Then market shares after one, two and three years are given by the **iterates**

$$\begin{aligned} \vec{\mathbf{y}}_1 &= A\vec{\mathbf{x}}, \\ \vec{\mathbf{y}}_2 &= A^2\vec{\mathbf{x}}, \\ \vec{\mathbf{y}}_3 &= A^3\vec{\mathbf{x}}. \end{aligned}$$

Fourier's replacement model gives succinct and useful formulas for the iterates: if $\vec{\mathbf{x}} = a_1\vec{\mathbf{v}}_1 + a_2\vec{\mathbf{v}}_2 + a_3\vec{\mathbf{v}}_3$, then

$$\begin{aligned} \vec{\mathbf{y}}_1 &= A\vec{\mathbf{x}} &&= a_1\lambda_1\vec{\mathbf{v}}_1 + a_2\lambda_2\vec{\mathbf{v}}_2 + a_3\lambda_3\vec{\mathbf{v}}_3, \\ \vec{\mathbf{y}}_2 &= A^2\vec{\mathbf{x}} &&= a_1\lambda_1^2\vec{\mathbf{v}}_1 + a_2\lambda_2^2\vec{\mathbf{v}}_2 + a_3\lambda_3^2\vec{\mathbf{v}}_3, \\ \vec{\mathbf{y}}_3 &= A^3\vec{\mathbf{x}} &&= a_1\lambda_1^3\vec{\mathbf{v}}_1 + a_2\lambda_2^3\vec{\mathbf{v}}_2 + a_3\lambda_3^3\vec{\mathbf{v}}_3. \end{aligned}$$

The advantage of Fourier's model is that an iterate A^n is computed directly, without computing the powers before it. Because $\lambda_1 = 1$ and

$\lim_{n\to\infty} |\lambda_2|^n = \lim_{n\to\infty} |\lambda_3|^n = 0$, then for large n

$$\vec{\mathbf{y}}_n \approx a_1(1)\vec{\mathbf{v}}_1 + a_2(0)\vec{\mathbf{v}}_2 + a_3(0)\vec{\mathbf{v}}_3 = \begin{pmatrix} a_1 \\ 5a_1/4 \\ 13a_1/12 \end{pmatrix}.$$

The numbers a_1, a_2, a_3 are related to x_1, x_2, x_3 by the equations $a_1 - a_2 - 4a_3 = x_1$, $5a_1/4 + 3a_3 = x_2$, $13a_1/12 + a_2 + a_3 = x_3$. Due to $x_1 + x_2 + x_3 = 1$, the value of a_1 is known, $a_1 = 3/10$. The three market shares after a long time period are therefore predicted to be $3/10$, $3/8$, $39/120$. The reader should verify the identity $\frac{3}{10} + \frac{3}{8} + \frac{39}{120} = 1$.

Stochastic Matrices

The special matrix A in (1) is a **stochastic matrix**, defined by the properties

$$\sum_{i=1}^{n} a_{ij} = 1, \quad a_{kj} \geq 0, \quad k, j = 1, \ldots, n.$$

The definition is memorized by the phrase *each column sum is one.* Stochastic matrices appear in **Leontief input-output models**, popularized by 1973 Nobel Prize economist Wassily Leontief.

Theorem 9 (Stochastic Matrix Properties)
Let A be a stochastic matrix. Then

(a) If $\vec{\mathbf{x}}$ is a vector with $x_1 + \cdots + x_n = 1$, then $\vec{\mathbf{y}} = A\vec{\mathbf{x}}$ satisfies $y_1 + \cdots + y_n = 1$.

(b) If $\vec{\mathbf{v}}$ is the sum of the columns of I, then $A^T\vec{\mathbf{v}} = \vec{\mathbf{v}}$. Therefore, $(1, \vec{\mathbf{v}})$ is an eigenpair of A^T.

(c) The characteristic equation $\det(A - \lambda I) = 0$ has a root $\lambda = 1$. All other roots satisfy $|\lambda| < 1$.

Proof of Stochastic Matrix Properties:
(a) $\sum_{i=1}^n y_i = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_j = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\right) x_j = \sum_{j=1}^n (1)x_j = 1$.

(b) Entry j of $A^T\vec{\mathbf{v}}$ is given by the sum $\sum_{i=1}^n a_{ij} = 1$.

(c) Apply **(b)** and the determinant rule $\det(B^T) = \det(B)$ with $B = A - \lambda I$ to obtain eigenvalue 1. Any other root λ of the characteristic equation has a corresponding eigenvector $\vec{\mathbf{x}}$ satisfying $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$. Let index j be selected such that $M = |x_j| > 0$ has largest magnitude. Then $\sum_{i\neq j} a_{ij}x_j + (a_{jj} - \lambda)x_j = 0$ implies $\lambda = \sum_{i=1}^n a_{ij}\dfrac{x_j}{M}$. Because $\sum_{i=1}^n a_{ij} = 1$, λ is a convex combination of n complex numbers $\{x_j/M\}_{j=1}^n$. These complex numbers are located in the unit disk, a convex set, therefore λ is located in the unit disk. By induction on n, motivated by the geometry for $n = 2$, it is argued that $|\lambda| = 1$ cannot happen for λ an eigenvalue different from 1 (details left to the reader). Therefore, $|\lambda| < 1$.

Details for the eigenpairs of (1): To be computed are the eigenvalues and eigenvectors for the 3×3 matrix

$$A = \frac{1}{10}\begin{pmatrix} 5 & 4 & 0 \\ 3 & 5 & 3 \\ 2 & 1 & 7 \end{pmatrix}.$$

Eigenvalues. The roots $\lambda = 1, 1/2, 1/5$ of the characteristic equation $\det(A - \lambda I) = 0$ are found by these details:

$$\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \begin{vmatrix} .5-\lambda & .4 & 0 \\ .3 & .5-\lambda & .3 \\ .2 & .1 & .7-\lambda \end{vmatrix} \\
&= \frac{1}{10} - \frac{8}{10}\lambda + \frac{17}{10}\lambda^2 - \lambda^3 && \text{Expand by cofactors.} \\
&= -\frac{1}{10}(\lambda-1)(2\lambda-1)(5\lambda-1) && \text{Factor the cubic.}
\end{aligned}$$

The factorization was found by long division of the cubic by $\lambda - 1$, the idea born from the fact that 1 is a root and therefore $\lambda - 1$ is a factor (the Factor Theorem of college algebra). An answer check in `maple`:

```
A:=(1/10)*Matrix([[5,4,0],[3,5,3],[2,1,7]]);
B:=A-lambda*Matrix([[1,0,0],[0,1,0],[0,0,1]]);
linalg[eigenvals](A); factor(linalg[det](B));
```

Eigenpairs. To each eigenvalue $\lambda = 1, 1/2, 1/5$ corresponds one **rref** calculation, to find the eigenvectors paired to λ. The three eigenvectors are given by (2). The details:

Eigenvalue $\lambda = 1$.

$$\begin{aligned}
A - (1)I &= \begin{pmatrix} .5-1 & .4 & 0 \\ .3 & .5-1 & .3 \\ .2 & .1 & .7-1 \end{pmatrix} \\
&\approx \begin{pmatrix} -5 & 4 & 0 \\ 3 & -5 & 3 \\ 2 & 1 & -3 \end{pmatrix} && \text{Multiply rule, multiplier=10.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 3 & -5 & 3 \\ 2 & 1 & -3 \end{pmatrix} && \text{Combination rule twice.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & -6 & 6 \\ 2 & 1 & -3 \end{pmatrix} && \text{Combination rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & -6 & 6 \\ 0 & 13 & -15 \end{pmatrix} && \text{Combination rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -\frac{12}{13} \\ 0 & 1 & -\frac{15}{13} \end{pmatrix} && \text{Multiply rule and combination rule.}
\end{aligned}$$

$$
\begin{aligned}
&\approx \begin{pmatrix} 1 & 0 & -\frac{12}{13} \\ 0 & 1 & -\frac{15}{13} \\ 0 & 0 & 0 \end{pmatrix} && \text{Swap rule.} \\
&= \mathbf{rref}(A-(1)I)
\end{aligned}
$$

An equivalent reduced echelon system is $x - 12z/13 = 0$, $y - 15z/13 = 0$. The free variable assignment is $z = t_1$ and then $x = 12t_1/13$, $y = 15t_1/13$. Let $x = 1$; then $t_1 = 13/12$. An eigenvector is given by $x = 1$, $y = 4/5$, $z = 13/12$.

Eigenvalue $\lambda = 1/2$.

$$
\begin{aligned}
A-(1/2)I &= \begin{pmatrix} .5-.5 & .4 & 0 \\ .3 & .5-.5 & .3 \\ .2 & .1 & .7-.5 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 4 & 0 \\ 3 & 0 & 3 \\ 2 & 1 & 2 \end{pmatrix} && \text{Multiply rule, factor=10.} \\
&\approx \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply rules.} \\
&= \mathbf{rref}(A-.5I)
\end{aligned}
$$

An eigenvector is found from the equivalent reduced echelon system $y = 0$, $x + z = 0$ to be $x = -1$, $y = 0$, $z = 1$.

Eigenvalue $\lambda = 1/5$.

$$
\begin{aligned}
A-(1/5)I &= \begin{pmatrix} .5-.2 & .4 & 0 \\ .3 & .5-.2 & .3 \\ .2 & .1 & .7-.2 \end{pmatrix} \\
&\approx \begin{pmatrix} 3 & 4 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 5 \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination rule.} \\
&= \mathbf{rref}(A-(1/5)I)
\end{aligned}
$$

An eigenvector is found from the equivalent reduced echelon system $x + 4z = 0$, $y - 3z = 0$ to be $x = -4$, $y = 3$, $z = 1$.

An answer check in `maple`:

```
A:=(1/10)*Matrix([[5,4,0],[3,5,3],[2,1,7]]);
linalg[eigenvects](A);
```

Coupled and Uncoupled Systems

The linear system of differential equations

$$\begin{aligned} x_1' &= -x_1 - x_3, \\ x_2' &= 4x_1 - x_2 - 3x_3, \\ x_3' &= 2x_1 - 4x_3, \end{aligned} \tag{3}$$

is called **coupled**, whereas the linear system of growth-decay equations

$$\begin{aligned} y_1' &= -3y_1, \\ y_2' &= -y_2, \\ y_3' &= -2y_3, \end{aligned} \tag{4}$$

is called **uncoupled**. The terminology *uncoupled* means that each differential equation in system (4) depends on exactly one variable, e.g., $y_1' = -3y_1$ depends only on variable y_1. In a *coupled* system, one of the differential equations must involve two or more variables.

Matrix characterization

Coupled system (3) and uncoupled system (4) can be written in matrix form, $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}' = D\vec{\mathbf{y}}$, with coefficient matrices

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 4 & -1 & -3 \\ 2 & 0 & -4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

If the coefficient matrix is **diagonal**, then the system is **uncoupled**. If the coefficient matrix is **not diagonal**, then one of the corresponding differential equations involves two or more variables and the system is called **coupled** or **cross-coupled**.

Solving Uncoupled Systems

An uncoupled system consists of independent growth-decay equations of the form $u' = au$. The solution formula $u = ce^{at}$ then leads to the general solution of the system of equations. For instance, system (4) has general solution

$$\begin{aligned} y_1 &= c_1 e^{-3t}, \\ y_2 &= c_2 e^{-t}, \\ y_3 &= c_3 e^{-2t}, \end{aligned} \tag{5}$$

where c_1, c_2, c_3 are **arbitrary constants**. The number of constants equals the dimension of the diagonal matrix D.

Coordinates and Coordinate Systems

If $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are three independent vectors in $\mathcal{R}^3$, then the matrix

$$P = \langle \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3 \rangle$$

is invertible. The columns $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ of P are called a **coordinate system**. The matrix P is called a **change of coordinates**.

Every vector $\vec{\mathbf{v}}$ in $\mathcal{R}^3$ can be uniquely expressed as

$$\vec{\mathbf{v}} = t_1\vec{\mathbf{v}}_1 + t_2\vec{\mathbf{v}}_2 + t_3\vec{\mathbf{v}}_3.$$

The values t_1, t_2, t_3 are called the **coordinates** of $\vec{\mathbf{v}}$ relative to the basis $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, or more succinctly, the **coordinates of $\vec{\mathbf{v}}$ relative to** P.

Viewpoint of a Driver

The physical meaning of a coordinate system $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ can be understood by considering an auto going up a mountain road. Choose orthogonal $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ to give positions in the driver's seat and define $\vec{\mathbf{v}}_3$ be the seat-back direction. These are **local coordinates** as viewed from the driver's seat. The road map coordinates x, y and the altitude z define the **global coordinates** for the auto's position $\vec{\mathbf{p}} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$.

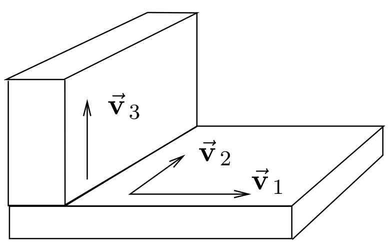


Figure 2. An auto seat.
The vectors $\vec{\mathbf{v}}_1(t)$, $\vec{\mathbf{v}}_2(t)$, $\vec{\mathbf{v}}_3(t)$ form an orthogonal triad which is a local coordinate system from the driver's viewpoint. The orthogonal triad changes continuously in t.

Change of Coordinates

A coordinate change from $\vec{\mathbf{y}}$ to $\vec{\mathbf{x}}$ is a linear algebraic equation $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ where the $n \times n$ matrix P is required to be invertible ($\det(P) \neq 0$). To illustrate, an instance of a change of coordinates from $\vec{\mathbf{y}}$ to $\vec{\mathbf{x}}$ is given by the linear equations

$$\text{(6)} \qquad \vec{\mathbf{x}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \vec{\mathbf{y}} \quad \text{or} \quad \left\{ \begin{array}{rcl} x_1 & = & y_1 + y_3, \\ x_2 & = & y_1 + y_2 - y_3, \\ x_3 & = & 2y_1 + y_3. \end{array} \right.$$

Constructing Coupled Systems

A general method exists to construct rich examples of coupled systems. The idea is to substitute a change of variables into a given uncoupled system. Consider a diagonal system $\vec{\mathbf{y}}' = D\vec{\mathbf{y}}$, like (4), and a change of variables $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$, like (6). Differential calculus applies to give

$$\begin{aligned} \vec{\mathbf{x}}' &= (P\vec{\mathbf{y}})' \\ &= P\vec{\mathbf{y}}' \\ &= PD\vec{\mathbf{y}} \\ &= PDP^{-1}\vec{\mathbf{x}}. \end{aligned} \tag{7}$$

The matrix $A = PDP^{-1}$ is *not triangular* in general, and therefore the change of variables produces a **cross-coupled** system.

An illustration. To give an example, substitute into uncoupled system (4) the change of variable equations (6). Use equation (7) to obtain

$$\vec{\mathbf{x}}' = \begin{pmatrix} -1 & 0 & -1 \\ 4 & -1 & -3 \\ 2 & 0 & -4 \end{pmatrix} \vec{\mathbf{x}} \quad \text{or} \quad \begin{cases} x_1' = -x_1 - x_3, \\ x_2' = 4x_1 - x_2 - 3x_3, \\ x_3' = 2x_1 - 4x_3. \end{cases} \tag{8}$$

This **cross-coupled** system (8) can be solved using relations (6), (5) and $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ to give the general solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-3t} \\ c_2 e^{-t} \\ c_3 e^{-2t} \end{pmatrix}. \tag{9}$$

Changing Coupled Systems to Uncoupled

We ask this question, motivated by the above calculations:

> Can every coupled system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ be subjected to a change of variables $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ which converts the system into a completely uncoupled system for variable $\vec{\mathbf{y}}(t)$?

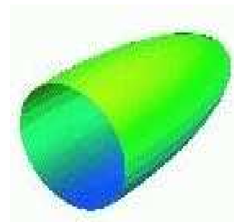
Under certain circumstances, this is true, and it leads to an elegant and especially simple expression for the general solution of the differential system, as in (9):

$$\vec{\mathbf{x}}(t) = P\vec{\mathbf{y}}(t).$$

The **task of eigenanalysis** is to simultaneously calculate from a cross-coupled system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ the change of variables $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ and the diagonal matrix D in the uncoupled system $\vec{\mathbf{y}}' = D\vec{\mathbf{y}}$

The **eigenanalysis coordinate system** is the set of n independent vectors extracted from the columns of P. In this coordinate system, the cross-coupled differential system (3) simplifies into a system of uncoupled growth-decay equations (4). Hence the terminology, *the method of simplifying coordinates.*

Eigenanalysis and Footballs



An ellipsoid or *football* is a geometric object described by its **semi-axes** (see Figure 3). In the vector representation, the **semi-axis directions** are unit vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ and the **semi-axis lengths** are the constants a, b, c. The vectors $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$, $c\vec{\mathbf{v}}_3$ form an **orthogonal triad**.

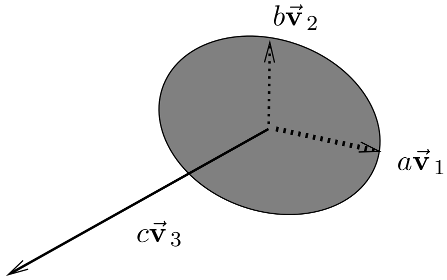


Figure 3. An American football.
An ellipsoid is built from orthonormal semi-axis directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ and the semi-axis lengths a, b, c. The semi-axis vectors are $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$, $c\vec{\mathbf{v}}_3$.

Two vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ are *orthogonal* if both are nonzero and their dot product $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is zero. Vectors are **orthonormal** if each has unit length and they are pairwise orthogonal. The orthogonal triad is an **invariant** of the ellipsoid's algebraic representations. Algebra does not change the triad: the invariants $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$, $c\vec{\mathbf{v}}_3$ must somehow be **hidden** in the equations that represent the football.

Algebraic eigenanalysis finds the hidden invariant triad $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$, $c\vec{\mathbf{v}}_3$ from the ellipsoid's algebraic equations. Suppose, for instance, that the equation of the ellipsoid is supplied as

$$x^2 + 4y^2 + xy + 4z^2 = 16.$$

A symmetric matrix A is constructed in order to write the equation in the form $\vec{\mathbf{X}}^T A \vec{\mathbf{X}} = 16$, where $\vec{\mathbf{X}}$ has components x, y, z. The replacement equation is[7]

$$(10) \qquad \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 16.$$

It is the 3×3 symmetric matrix A in (10) that is subjected to algebraic eigenanalysis. The matrix calculation will compute the unit semi-axis directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$, called the **hidden vectors** or **eigenvectors**. The semi-axis lengths a, b, c are computed at the same time, by finding the **hidden values**[8] or **eigenvalues** λ_1, λ_2, λ_3, known to satisfy the

[7] The reader should pause here and multiply matrices in order to verify this statement. Halving of the entries corresponding to cross-terms generalizes to any ellipsoid.

[8] The terminology *hidden* arises because neither the semi-axis lengths nor the semi-axis directions are revealed directly by the ellipsoid equation.

relations

$$\lambda_1 = \frac{16}{a^2}, \quad \lambda_2 = \frac{16}{b^2}, \quad \lambda_3 = \frac{16}{c^2}.$$

For the illustration, the football dimensions are $a = 2$, $b = 1.98$, $c = 4.17$. Details of the computation are delayed until page 666.

The Ellipse and Eigenanalysis

An ellipse equation in **standard form** is $\lambda_1 x^2 + \lambda_2 y^2 = 1$, where $\lambda_1 = 1/a^2$, $\lambda_2 = 1/b^2$ are expressed in terms of the semi-axis lengths a, b. The expression $\lambda_1 x^2 + \lambda_2 y^2$ is called a **quadratic form**. The study of the ellipse $\lambda_1 x^2 + \lambda_2 y^2 = 1$ is equivalent to the study of the quadratic form equation

$$\vec{\mathbf{r}}^T D\vec{\mathbf{r}} = 1, \quad \text{where} \quad \vec{\mathbf{r}} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Cross-terms. An ellipse may be represented by an equation in a uv-coordinate system having a cross-term uv, e.g., $4u^2+8uv+10v^2 = 5$. The expression $4u^2 + 8uv + 10v^2$ is again called a quadratic form. Calculus courses provide methods to eliminate the cross-term and represent the equation in standard form, by a **rotation**

$$\begin{pmatrix} u \\ v \end{pmatrix} = R\begin{pmatrix} x \\ y \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The angle θ in the rotation matrix R represents the rotation of uv-coordinates into standard xy-coordinates.

Eigenanalysis computes angle θ through the columns of R, which are the unit semi-axis directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ for the ellipse $4u^2 + 8uv + 10v^2 = 5$. If the quadratic form $4u^2 + 8uv + 10v^2$ is represented as $\vec{\mathbf{r}}^T A\,\vec{\mathbf{r}}$, then

$$\vec{\mathbf{r}} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}, \quad R = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$$

$$\lambda_1 = 12, \quad \vec{\mathbf{v}}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda_2 = 2, \quad \vec{\mathbf{v}}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Rotation matrix angle θ. The components of eigenvector $\vec{\mathbf{v}}_1$ can be used to determine $\theta = -63.4°$:

$$\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{cases} \cos\theta & = \frac{1}{\sqrt{5}}, \\ -\sin\theta & = \frac{2}{\sqrt{5}}. \end{cases}$$

The interpretation of angle θ: rotate the orthonormal basis $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ by angle $\theta = -63.4°$ in order to obtain the standard unit basis vectors $\vec{\mathbf{i}}$,

$\vec{\mathbf{j}}$. Most calculus texts discuss only the inverse rotation, where x, y are given in terms of u, v. In these references, θ is the negative of the value given here, due to a different geometric viewpoint.[9]

Semi-axis lengths. The lengths $a \approx 1.55$, $b \approx 0.63$ for the ellipse $4u^2+8uv+10v^2=5$ are computed from the eigenvalues $\lambda_1 = 12$, $\lambda_2 = 2$ of matrix A by the equations

$$\frac{\lambda_1}{5} = \frac{1}{a^2}, \quad \frac{\lambda_2}{5} = \frac{1}{b^2}.$$

Geometry. The ellipse $4u^2 + 8uv + 10v^2 = 5$ is completely determined by the orthogonal semi-axis vectors $a\vec{\mathbf{v}}_1$, $b\vec{\mathbf{v}}_2$. The rotation R is a rigid motion which maps these vectors into $a\vec{\imath}$, $b\vec{\jmath}$, where $\vec{\imath}$ and $\vec{\jmath}$ are the standard unit vectors in the plane.

The θ-rotation R maps $4u^2+8uv+10v^2 = 5$ into the xy-equation $\lambda_1 x^2 + \lambda_2 y^2 = 5$, where λ_1, λ_2 are the eigenvalues of A. To see why, let $\vec{\mathbf{r}} = R\vec{\mathbf{s}}$ where $\vec{\mathbf{s}} = \begin{pmatrix} x & y \end{pmatrix}^T$. Then $\vec{\mathbf{r}}^T A \vec{\mathbf{r}} = \vec{\mathbf{s}}^T (R^T A R)\vec{\mathbf{s}}$. Using $R^T R = I$ gives $R^{-1} = R^T$ and $R^T A R = \mathbf{diag}(\lambda_1, \lambda_2)$. Finally, $\vec{\mathbf{r}}^T A \vec{\mathbf{r}} = \lambda_1 x^2 + \lambda_2 y^2$.

Orthogonal Triad Computation

Let's compute the semiaxis directions $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ for the ellipsoid $x^2 + 4y^2 + xy + 4z^2 = 16$. To be applied is Theorem 4. As explained on page 664, the starting point is to represent the ellipsoid equation as a quadratic form $X^T A X = 16$, where the symmetric matrix A is defined by

$$A = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

College algebra. The **characteristic polynomial** $\det(A - \lambda I) = 0$ determines the eigenvalues or hidden values of the matrix A. By cofactor expansion, this polynomial equation is

$$(4-\lambda)((1-\lambda)(4-\lambda) - 1/4) = 0$$

with roots 4, $5/2 + \sqrt{10}/2$, $5/2 - \sqrt{10}/2$.

[9] Rod Serling, author and playwright for *The Twilight Zone*, enjoyed the view from the other side of the mirror.

Eigenpairs. It will be shown that three eigenpairs are

$$\lambda_1 = 4, \quad \vec{\mathbf{x}}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$\lambda_2 = \frac{5+\sqrt{10}}{2}, \quad \vec{\mathbf{x}}_2 = \begin{pmatrix} \sqrt{10}-3 \\ 1 \\ 0 \end{pmatrix},$$
$$\lambda_3 = \frac{5-\sqrt{10}}{2}, \quad \vec{\mathbf{x}}_3 = \begin{pmatrix} \sqrt{10}+3 \\ -1 \\ 0 \end{pmatrix}.$$

The vector norms of the eigenvectors are given by $\|\vec{\mathbf{x}}_1\| = 1$, $\|\vec{\mathbf{x}}_2\| = \sqrt{20+6\sqrt{10}}$, $\|\vec{\mathbf{x}}_3\| = \sqrt{20-6\sqrt{10}}$. The orthonormal semi-axis directions $\vec{\mathbf{v}}_k = \vec{\mathbf{x}}_k/\|\vec{\mathbf{x}}_k\|$, $k = 1,2,3$, are then given by the formulas

$$\vec{\mathbf{v}}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{pmatrix} \frac{\sqrt{10}-3}{\sqrt{20-6\sqrt{10}}} \\ \frac{1}{\sqrt{20-6\sqrt{10}}} \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{pmatrix} \frac{\sqrt{10}+3}{\sqrt{20+6\sqrt{10}}} \\ \frac{-1}{\sqrt{20+6\sqrt{10}}} \\ 0 \end{pmatrix}.$$

Toolkit sequence details.

$$\langle A - \lambda_1 I, \vec{\mathbf{0}}\rangle = \left(\begin{array}{ccc|c} 1-4 & 1/2 & 0 & 0 \\ 1/2 & 4-4 & 0 & 0 \\ 0 & 0 & 4-4 & 0 \end{array}\right)$$
$$\approx \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{Used combination, multiply and swap rules. Found } \textbf{rref}.$$

$$\langle A - \lambda_2 I, \vec{\mathbf{0}}\rangle = \left(\begin{array}{ccc|c} \frac{-3-\sqrt{10}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3-\sqrt{10}}{2} & 0 & 0 \\ 0 & 0 & \frac{3-\sqrt{10}}{2} & 0 \end{array}\right)$$
$$\approx \left(\begin{array}{ccc|c} 1 & 3-\sqrt{10} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{All three rules.}$$

$$\langle A - \lambda_3 I, \vec{\mathbf{0}}\rangle = \left(\begin{array}{ccc|c} \frac{-3+\sqrt{10}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{3+\sqrt{10}}{2} & 0 & 0 \\ 0 & 0 & \frac{3+\sqrt{10}}{2} & 0 \end{array}\right)$$
$$\approx \left(\begin{array}{ccc|c} 1 & 3+\sqrt{10} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{All three rules.}$$

Solving the corresponding reduced echelon systems gives the preceding formulas for the eigenvectors $\vec{\mathbf{x}}_1$, $\vec{\mathbf{x}}_2$, $\vec{\mathbf{x}}_3$. The equation for the ellipsoid is $\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 16$, where the multipliers of the square terms are the eigenvalues of A and X, Y, Z define the new coordinate system determined by the eigenvectors of A. This equation can be re-written in the form $X^2/a^2 + Y^2/b^2 + Z^2/c^2 = 1$, provided the semi-axis lengths a, b, c are defined by the relations $a^2 = 16/\lambda_1$, $b^2 = 16/\lambda_2$, $c^2 = 16/\lambda_3$. After computation, $a = 2$, $b = 1.98$, $c = 4.17$.