Algebraic Eigenanalysis

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The Matrix Eigenanalysis Method

The preceding discussion of data conversion now gives way to synthetic abstract definitions which distill the essential theory of eigenanalysis.

All of this is algebra, devoid of motivation or application.

Eigenpairs

Definition 1 (Eigenpair)
A pair $(\lambda, \vec{\mathrm{v}})$, where $\vec{\mathrm{v}} \neq \vec{0}$ is a vector and λ is a complex number, is called an **eigenpair** of the $n \times n$ matrix A provided

(1) $$A\vec{\mathrm{v}} = \lambda\vec{\mathrm{v}} \quad (\vec{\mathrm{v}} \neq \vec{0} \text{ required}).$$

- The **nonzero** requirement in (1) results from seeking directions for a coordinate system: the zero vector is not a direction.
- Any vector $\vec{\mathrm{v}} \neq \vec{0}$ that satisfies (1) is called an **eigenvector** for λ and the value λ is called an **eigenvalue** of the square matrix A.

Eigenanalysis Algorithm

Theorem 1 (Algebraic Eigenanalysis)
Eigenpairs $(\lambda, \vec{v})$ of an $n \times n$ matrix A are found by this two-step algorithm:

Step 1 (College Algebra). Solve for eigenvalues λ in the nth order polynomial equation $\det(A - \lambda I) = 0$.

Step 2 (Linear Algebra). For a given root λ from **Step 1**, a corresponding eigenvector $\vec{v} \neq \vec{0}$ is found by a sequence of toolkit operations[a] to the homogeneous linear equation

$$(A - \lambda I)\vec{v} = \vec{0}.$$

The answer for $\vec{v}$ is the list of partial derivatives $\partial_{t_1}\vec{v}$, $\partial_{t_2}\vec{v}$, ..., where t_1, t_2, ... are invented symbols assigned to the free variables. In linear algebra, these vectors are called *Strang's Special Solutions*.

The reader is asked to apply the algorithm to the identity matrix I; then **Step 1** gives n duplicate answers $\lambda = 1$ and **Step 2** gives n answers, the columns of the identity matrix I.

[a] For $B\vec{v} = \vec{0}$, the sequence begins with B, instead of augmenting zero to B. The sequence ends with $\mathrm{rref}(B)$. Then the scalar reduced echelon system is written, followed by assignment of free variables and display of the vector general solution $\vec{v}$.

Proof of the Algebraic Eigenanalysis Theorem

The equation $A\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}$ is equivalent to $(A - \lambda I)\vec{\mathbf{v}} = \vec{\mathbf{0}}$, which is a set of homogeneous equations, consistent always because of the solution $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Fix λ and define $B = A - \lambda I$. We show that an eigenpair $(\lambda, \vec{\mathbf{v}})$ exists with $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$ if and only if $\det(B) = 0$, i.e., $\det(A - \lambda I) = 0$. There is a unique solution $\vec{\mathbf{v}}$ to the homogeneous equation $B\vec{\mathbf{v}} = \vec{\mathbf{0}}$ exactly when Cramer's rule applies, in which case $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is the unique solution. All that Cramer's rule requires is $\det(B) \neq 0$. Therefore, an eigenpair exists exactly when Cramer's rule fails to apply, which is when the determinant of coefficients is zero: $\det(B) = 0$.

Eigenvectors for λ are found from the general solution to the system of equations $B\vec{\mathbf{v}} = 0$ where $B = A - \lambda I$. The $\mathbf{rref}$ method produces systematically a reduced echelon system from which the general solution $\vec{\mathbf{v}}$ is written, depending on invented symbols $t_1, \ldots, t_k$. Since there is never a unique solution, at least one free variable exists. In particular, the last step $\mathbf{rref}(B)$ of the sequence has a row of zeros, which is a useful sanity test.

The **basis of eigenvectors** for λ is obtained from the general solution $\vec{\mathbf{v}}$, which is a linear combination involving the parameters $t_1, \ldots, t_k$. The **basis elements** are reported as the list of partial derivatives $\partial_{t_1}\vec{\mathbf{v}}, \ldots, \partial_{t_k}\vec{\mathbf{v}}$.

Eigenpair Packages

The eigenpairs of a 3×3 matrix for which Fourier's model holds are labeled

$$(\lambda_1, \vec{v}_1), \quad (\lambda_2, \vec{v}_2), \quad (\lambda_3, \vec{v}_3).$$

An **eigenvector package** is a matrix package P of eigenvectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ given by

$$P = \text{aug}(\vec{v}_1, \vec{v}_2, \vec{v}_3).$$

An **eigenvalue package** is a matrix package D of eigenvalues given by

$$D = \mathbf{diag}(\lambda_1, \lambda_2, \lambda_3).$$

Important is the *pairing* that is inherited from the eigenpairs, which dictates the packaging order of the eigenvectors and eigenvalues. Matrices P, D are **not unique**: possible are $3!$ $(= 6)$ column permutations.

Data Conversion Example

The eigenvalues for the $\mathbf{3 \times 3}$ data conversion problem are $\boldsymbol{\lambda_1 = 1}$, $\boldsymbol{\lambda_2 = 0.001}$, $\boldsymbol{\lambda_3 = 0.01}$ and the eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are the columns of the identity matrix $\boldsymbol{I}$. Then the eigenpair packages are

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.01 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 2 (Eigenpair Packages)

Let $\boldsymbol{P}$ be a matrix package of eigenvectors and $\boldsymbol{D}$ the corresponding matrix package of eigenvalues. Then for all vectors $\vec{\mathbf{c}}$,

$$\boldsymbol{AP\vec{\mathbf{c}} = PD\vec{\mathbf{c}}}.$$

Proof of the Eigenpair Package Theorem

Proof: The result is valid for $n \times n$ matrices.
We prove the eigenpair package theorem for 3×3 matrices. The two sides of the equation are expanded as follows.

$$
\begin{aligned}
P D \vec{c} &= P \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} && \text{Expand RHS.} \\
&= P \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \lambda_3 c_3 \end{pmatrix} \\
&= \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2 + \lambda_3 c_3 \vec{v}_3 && \text{Because } P \text{ has columns } \vec{v}_1, \vec{v}_2, \vec{v}_3. \\
A P \vec{c} &= A(c_1 \vec{v}_2 + c_2 \vec{v}_2 + c_3 \vec{v}_3) && \text{Expand LHS.} \\
&= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + c_3 \lambda_3 \vec{v}_3 && \text{Fourier's model.}
\end{aligned}
$$

The Equation $AP = PD$

The question of Fourier's model holding for a given 3×3 matrix A is settled here. According to the result, a matrix A for which Fourier's model holds can be constructed by the formula $A = PDP^{-1}$ where D is any diagonal matrix and P is an invertible matrix.

Theorem 3 ($AP = PD$)
Fourier's model $A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + c_3\lambda_3\vec{v}_3$ holds for eigenpairs $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$, $(\lambda_3, \vec{v}_3)$ if and only if the packages

$$P = \text{aug}(\vec{v}_1, \vec{v}_2, \vec{v}_3), \quad D = \textbf{diag}(\lambda_1, \lambda_2, \lambda_3)$$

satisfy the two requirements

1. Matrix P is invertible, e.g., $\det(P) \neq 0$.
2. Matrix $A = PDP^{-1}$, or equivalently, $AP = PD$.

Proof Details for $AP = PD$

Assume Fourier's model holds. Define P and D to be the eigenpair packages. Then **1** holds, because the columns of P are independent. By Theorem 2, $AP\vec{c} = PD\vec{c}$ for all vectors $\vec{c}$. Taking $\vec{c}$ equal to a column of the identity matrix I implies the columns of AP and PD are identical, that is, $AP = PD$. A multiplication of $AP = PD$ by P^{-1} gives **2**.

Conversely, let P and D be given packages satisfying **1**, **2**. Define $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ to be the columns of P. Then the columns pass the rank test, because P is invertible, proving independence of the columns. Define λ_1, λ_2, λ_3 to be the diagonal elements of D. Using $AP = PD$, we calculate the two sides of $AP\vec{c} = PD\vec{c}$ as in the proof of Theorem 2, which shows that $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_2\vec{v}_3$ implies $A\vec{x} = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + c_3\lambda_3\vec{v}_3$. Hence Fourier's model holds.

Diagonalization

A square matrix $\boldsymbol{A}$ is called **diagonalizable** provided $\boldsymbol{AP} = \boldsymbol{PD}$ for some diagonal matrix $\boldsymbol{D}$ and invertible matrix $\boldsymbol{P}$. The preceding discussions imply that $\boldsymbol{D}$ must be a package of eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{P}$ must be the corresponding package of eigenvectors of $\boldsymbol{A}$. The requirement on $\boldsymbol{P}$ to be invertible is equivalent to asking that the eigenvectors of $\boldsymbol{A}$ be independent and equal in number to the column dimension of $\boldsymbol{A}$.

The matrices $\boldsymbol{A}$ for which Fourier's model is valid is precisely the class of diagonalizable matrices. This class is not the set of all square matrices: there are matrices $\boldsymbol{A}$ for which Fourier's model is invalid. They are called **non-diagonalizable matrices**.

Distinct Eigenvalues and Diagonalization

The construction for eigenvector package P always produces independent columns. Even if A has fewer than n eigenpairs, the construction still produces independent eigenvectors. In such **non-diagonalizable** cases, caused by insufficient columns to form P, matrix A must have an eigenvalue of multiplicity greater than one.

If all eigenvalues are distinct, then the correct number of independent eigenvectors were found and A is then **diagonalizable** with packages D, P satisfying $AP = PD$. This proves the following result.

Theorem 4 (Distinct Eigenvalues)

If an $n \times n$ matrix A has n distinct eigenvalues, then it has n eigenpairs and A is diagonalizable with eigenpair packages D, P satisfying $AP = PD$.

1 Example (Computing 2×2 Eigenpairs)

Find all eigenpairs of the 2×2 matrix $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

Solution:

College Algebra. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= (1-\lambda)(-1-\lambda) && \text{Sarrus' rule.} \end{aligned}$$

Solution:

Linear Algebra. The eigenpairs are $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(-1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. Details:

Eigenvector for $\lambda_1 = 1$.

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} 1-\lambda_1 & 0 \\ 2 & -1-\lambda_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} && \text{Swap and multiply rules.} \\ &= \text{rref}(A - \lambda_1 I) && \text{Reduced echelon form.} \end{aligned}$$

The partial derivative $\partial_{t_1}\vec{v}$ of the general solution $x = t_1$, $y = t_1$ is eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = -1$.

$$\begin{aligned} A - \lambda_2 I &= \begin{pmatrix} 1-\lambda_2 & 0 \\ 2 & -1-\lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\ &= \text{rref}(A - \lambda_2 I) && \text{Reduced echelon form.} \end{aligned}$$

The partial derivative $\partial_{t_1}\vec{v}$ of the general solution $x = 0$, $y = t_1$ is eigenvector $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2 Example (Computing 3×3 Eigenpairs)

Find all eigenpairs of the 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

College Algebra

The eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$, $\lambda_3 = 3$. Details:

$$
\begin{aligned}
0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\
&= \begin{vmatrix} 1-\lambda & 2 & 0 \\ -2 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\
&= ((1-\lambda)^2 + 4)(3-\lambda) && \text{Cofactor rule and Sarrus' rule.}
\end{aligned}
$$

Root $\lambda = 3$ is found from the factored form above. The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1-\lambda)^2 + 4 = 0$. Alternatively, take roots across $(\lambda - 1)^2 = -4$.

Linear Algebra

The eigenpairs are

$$\left(1+2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right), \left(1-2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Details appear below.

Eigenvector $\vec{\mathrm{v}}_1$ for $\lambda_1 = 1 + 2i$

$$
\begin{aligned}
B &= A - \lambda_1 I \\
&= \begin{pmatrix} 1-\lambda_1 & 2 & 0 \\ -2 & 1-\lambda_1 & 0 \\ 0 & 0 & 3-\lambda_1 \end{pmatrix} \\
&= \begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \\ 0 & 0 & 2-2i \end{pmatrix} \\
&\approx \begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Combination, factor}{=}{-i}. \\
&\approx \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \text{Swap rule.} \\
&= \mathrm{rref}(A - \lambda_1 I) && \text{Reduced echelon form.}
\end{aligned}
$$

The partial derivative $\partial_{t_1}\vec{\mathrm{v}}$ of the general solution $x = -it_1$, $y = t_1$, $z = 0$ is eigenvector $\vec{\mathrm{v}}_1 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Eigenvector $\vec{\mathbf{v}}_2$ for $\boldsymbol{\lambda_2 = 1 - 2i}$

The problem $(\boldsymbol{A} - \boldsymbol{\lambda}_2\boldsymbol{I})\vec{\mathbf{v}}_2 = \vec{\mathbf{0}}$ has solution $\vec{\mathbf{v}}_2 = \overline{\vec{\mathbf{v}}_1}$.

To see why, take conjugates across the equation to give $(\overline{\boldsymbol{A}} - \overline{\boldsymbol{\lambda}_2}\boldsymbol{I})\overline{\vec{\mathbf{v}}_2} = \vec{\mathbf{0}}$. Then $\overline{\boldsymbol{A}} = \boldsymbol{A}$ ($\boldsymbol{A}$ is real) and $\boldsymbol{\lambda}_1 = \overline{\boldsymbol{\lambda}_2}$ gives $(\boldsymbol{A} - \boldsymbol{\lambda}_1\boldsymbol{I})\overline{\vec{\mathbf{v}}_2} = \mathbf{0}$. Then $\overline{\vec{\mathbf{v}}_2} = \vec{\mathbf{v}}_1$.

Finally,

$$\vec{\mathbf{v}}_2 = \overline{\overline{\vec{\mathbf{v}}_2}} = \overline{\vec{\mathbf{v}}_1} = \begin{pmatrix} \boldsymbol{i} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}.$$

Eigenvector v_3 for $\lambda_3 = 3$

$$\begin{aligned} A - \lambda_3 I &= \begin{pmatrix} 1-\lambda_3 & 2 & 0 \\ -2 & 1-\lambda_3 & 0 \\ 0 & 0 & 3-\lambda_3 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Multiply rule.} \\ &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\ &= \operatorname{rref}(A - \lambda_3 I) && \text{Reduced echelon form.} \end{aligned}$$

The partial derivative $\partial_{t_1}\vec{\mathbf{v}}$ of the general solution $x = 0$, $y = 0$, $z = t_1$ is eigenvector

$$\vec{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$