

Elementary Matrices and Toolkit Sequences

- Elementary Matrices
 - Definition of elementary matrix
 - Computer algebra systems and elementary matrices
 - Constructing elementary matrices E and their inverses E^{-1}
- Fundamental Theorem on Elementary Matrices
- A certain 6-frame toolkit sequence.
- Toolkit Sequence Details
- Fundamental Theorem Illustrated
- The RREF Inverse Method

Elementary Matrices

Definition. An elementary matrix E is the result of applying a combination, multiply or swap rule to the identity matrix.

An elementary matrix is then the **second frame** after a `combo`, `swap` or `mult` toolkit operation which has been applied to a **first frame** equal to the identity matrix.

Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

First frame = identity matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$$

Second frame

Elementary `combo` matrix

`combo(1, 3, -5)`

Computer algebra systems and elementary matrices

The computer algebra system maple displays typical 4×4 elementary matrices (C=Combination, M=Multiply, S=Swap) as follows.

<code>with(linalg):</code>	<code>with(LinearAlgebra):</code>
<code>Id:=diag(1,1,1,1);</code>	<code>Id:=IdentityMatrix(4);</code>
<code>C:=addrow(Id,2,3,c);</code>	<code>C:=RowOperation(Id,[3,2],c);</code>
<code>M:=mulrow(Id,3,m);</code>	<code>M:=RowOperation(Id,3,m);</code>
<code>S:=swaprow(Id,1,4);</code>	<code>S:=RowOperation(Id,[4,1]);</code>

The answers:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Constructing elementary matrices E and their inverses E^{-1} _____

Mult Change a one in the identity matrix to symbol $m \neq 0$.

Combo Change a zero in the identity matrix to symbol c .

Swap Interchange two rows of the identity matrix.

Constructing E^{-1} from elementary matrix E

Mult Change diagonal multiplier $m \neq 0$ in E to $1/m$.

Combo Change multiplier c in E to $-c$.

Swap The inverse of E is E itself.

Fundamental Theorem on Elementary Matrices

Theorem 1 (Toolkit sequences and elementary matrices)

In a Toolkit sequence, let the second frame A_2 be obtained from the first frame A_1 by a `combo`, `swap` or `mult` toolkit operation. Let n equal the row dimension of A_1 . Then there is correspondingly an $n \times n$ `combo`, `swap` or `mult` elementary matrix E such that

$$A_2 = EA_1.$$

Theorem 2 (The `rref` and elementary matrices)

Let A be a given matrix of row dimension n . Then there exist $n \times n$ elementary matrices $E_1, E_2, \dots, E_k$ such that

$$\text{rref}(A) = E_k \cdots E_2 E_1 A.$$

Proof of Theorem 1

The first result is the observation that left multiplication of matrix A_1 by elementary matrix E gives the answer $A_2 = EA_1$ which is obtained by applying the corresponding `combo`, `swap` or `mult` toolkit operation. This fact is discovered by doing examples, then a formal proof can be constructed (not presented here).

Proof of Theorem 2

The second result applies the first result multiple times to obtain elementary matrices $E_1, E_2, \dots$ which represent the multiply, combination and swap operations performed in the toolkit sequence which take the First Frame $A_1 = A$ into the Last Frame $A_{k+1} = \text{rref}(A_1)$. Combining the identities

$$A_2 = E_1 A_1, \quad A_3 = E_2 A_2, \quad \dots, \quad A_{k+1} = E_k A_k$$

gives the matrix multiply equation

$$A_{k+1} = E_k E_{k-1} \cdots E_2 E_1 A_1$$

or equivalently the theorem's result, because $A_{k+1} = \text{rref}(A)$ and $A_1 = A$.

A certain 6-frame toolkit sequence _____.

$$A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 1, original matrix.}$$

$$A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 2, combo(1,2,-2).}$$

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 3 \end{pmatrix} \quad \text{Frame 3, mult(2,-1/6).}$$

$$A_4 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{pmatrix} \quad \text{Frame 4, combo(1,3,-3).}$$

$$A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Frame 5, combo(2,3,-6).}$$

$$A_6 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Frame 6, combo(2,1,-3). Found rref}(A_1).$$

Continued

The corresponding 3×3 elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 2, combo(1,2,-2) applied to } I.$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 3, mult(2,-1/6) applied to } I.$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad \text{Frame 4, combo(1,3,-3) applied to } I.$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \quad \text{Frame 5, combo(2,3,-6) applied to } I.$$

$$E_5 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{Frame 6, combo(2,1,-3) applied to } I.$$

Frame Sequence Details

$$A_2 = E_1 A_1$$

Frame 2, E_1 equals $\text{combo}(1,2,-2)$ on I .

$$A_3 = E_2 A_2$$

Frame 3, E_2 equals $\text{mult}(2,-1/6)$ on I .

$$A_4 = E_3 A_3$$

Frame 4, E_3 equals $\text{combo}(1,3,-3)$ on I .

$$A_5 = E_4 A_4$$

Frame 5, E_4 equals $\text{combo}(2,3,-6)$ on I .

$$A_6 = E_5 A_5$$

Frame 6, E_5 equals $\text{combo}(2,1,-3)$ on I .

$$A_6 = E_5 E_4 E_3 E_2 E_1 A_1$$

Summary frames 1-6.

Then

$$\text{rref}(A_1) = E_5 E_4 E_3 E_2 E_1 A_1,$$

which is the result of the Theorem.

Fundamental Theorem Illustrated

The summary:

$$A_6 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_1$$

Because $A_6 = \text{rref}(A_1)$, the above equation gives the inverse relationship

$$A_1 = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} \text{rref}(A_1).$$

Each inverse matrix is simplified by the rules for constructing E^{-1} from elementary matrix E , the result being

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{rref}(A_1)$$

Theorem 3 (RREF Inverse Method)

$$\text{rref}(\langle A|I \rangle) = \langle I|B \rangle \quad \text{if and only if} \quad AB = I.$$

Proof: For *any* matrix E there is the matrix multiply identity

$$E\langle C|D \rangle = \langle EC|ED \rangle.$$

This identity is proved by arguing that each side has identical columns. For example, $\text{col}(\text{LHS}, 1) = E \text{col}(C, 1) = \text{col}(\text{RHS}, 1)$.

Assume $C = \langle A|I \rangle$ satisfies $\text{rref}(C) = \langle I|B \rangle$. The fundamental theorem of elementary matrices implies $E_k \cdots E_1 C = \text{rref}(C)$. Then

$$\text{rref}(C) = \langle E_k \cdots E_1 A \mid E_k \cdots E_1 I \rangle = \langle I|B \rangle$$

implies that $E_k \cdots E_1 A = I$ and $E_k \cdots E_1 I = B$. Together, $BA = I$ and then B is the inverse of A .

Conversely, assume that $AB = I$. Then A has inverse B . The fundamental theorem of elementary matrices implies the identity $E_k \cdots E_1 A = \text{rref}(A) = I$. It follows that $B = E_k \cdots E_1$. Then $\text{rref}(C) = E_k \cdots E_1 \langle A|I \rangle = \langle E_k \cdots E_1 A \mid E_k \cdots E_1 I \rangle = \langle I|B \rangle$.