Rank, Nullity, Dimension and Elimination

- Last Frame Test
- Last Frame Example
- Definition: Reduced Echelon System
- Definition: Rank and Nullity
- Definition: Dimension
- Theorem: Rank plus Nullity equals Variable Count
- Elimination
  - Theorem: Elimination
  - An Elimination Algorithm
  - Example

Last Frame Test

In a sequence of combo-swap-mult operations, a frame without a signal equation $\mathbf{0 = 1}$ is called the *Last Frame* provided:

- Each nonzero equation has a lead variable. This means the variable has leading coefficient 1 and it appears nowhere else in the system.
- Equations $\mathbf{0 = 0}$ appear at the end.
- Variables within an equation appear in variable list order.
- Nonzero equations are listed with lead variables in variable list order.

Last Frame Example

Let's assume variable list order $\boldsymbol{x, y, z}$.

$$\left\{\begin{array}{rcrcrcl} \boldsymbol{x} & + & \boldsymbol{y} & & & = & \boldsymbol{2,} \\ & & & & \boldsymbol{z} & = & \boldsymbol{3,} \\ & & & & \boldsymbol{0} & = & \boldsymbol{0.} \end{array}\right.$$

The lead variables are $\boldsymbol{x}$, $\boldsymbol{z}$ and the free variable is $\boldsymbol{y}$.

Definition 1 (Reduced Echelon System)
A linear system which passes the last frame test is called a **reduced echelon system**.

Definition 2 (Rank and Nullity)
Assume the last frame test has been passed. Then

- Rank = number of lead variables,
- Nullity = number of free variables (non-lead variables).

Determining the rank and nullity of a system

- Display a frame sequence whose first frame is that system, and whose last frame passes the *Last Frame Test*. The last frame is a *reduced echelon system*.
- The rank of the system is the number of lead variables in the last frame.
- The nullity of the system is the number of variables minus the rank. It is exactly the free variable count.

Theorem 1 (Rank and Nullity)
The following equations hold:

$$\begin{aligned}\textbf{lead variable count} + \textbf{free variable count} &= \text{number of variables}, \\ \text{or} & \\ \textbf{rank} + \textbf{nullity} &= \text{number of variables}.\end{aligned}$$

Definition 3 (Dimension)
The **dimension** of a system of equations is the number of free variables, which is the number of invented symbols assigned in the Last Frame Algorithm.

> The case of a homogeneous $\mathbf{3} \times \mathbf{3}$ system gives geometric meaning to the term **dimension**. In this case, dimension 0 is the unique solution case, geometrically a point, and dimensions 1, 2, 3 enumerate the possibilities for the general solution to describe a line, plane or all of space.
>
> Dimension is devoid of geometric meaning. It is a gross measure of the complexity of the solution set of the system of equations.

Elimination

The elimination algorithm applies at each algebraic step one of the three toolkit rules **swap**, **multiply** and **combination**.

- The objective of each algebraic step is to **increase the number of lead variables**. The process stops when a signal equation (typically $\mathbf{0 = 1}$) is found. Otherwise, it stops when no more lead variables can be found, and then the last system of equations is a **reduced echelon system**. A detailed explanation of the process appears *infra*.
- Reversibility of the algebraic steps means that no solutions are created nor destroyed throughout the algebraic steps: the original system and all systems in the intermediate steps have *exactly the same solutions*.
- The final reduced echelon system has either a unique solution or infinitely many solutions. In both cases we report the **general solution**. In the infinitely many solution case, the **last frame algorithm** is used to write out a general solution.

Theorem 2 (Elimination)
Every linear system has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated application of the toolkit rules **swap**, **multiply** and **combination**.

An Elimination Algorithm

An equation is said to be **processed** if it has a lead variable. Otherwise, the equation is said to be **unprocessed**.

1. If an equation "$0 = 0$" appears, then move it to the end. If a signal equation "$0 = c$" appears ($c \neq 0$ required), then the system is inconsistent. In this case, the algorithm halts and we report **no solution**.
2. Identify the **first symbol** x_r, in variable list order $x_1, \ldots, x_n$, which appears in some unprocessed equation. Apply the **multiply** rule to insure x_r has leading coefficient one. Apply the **combination** rule to eliminate variable x_r from all other equations. Then x_r is a **lead variable**: the number of lead variables has been increased by one.
3. Apply the **swap** rule repeatedly to move this equation past all processed equations, but before the unprocessed equations. Mark the equation as **processed**, e.g., replace x_r by boxed symbol $\boxed{x_r}$.
4. Repeat steps 1–3, until all equations have been processed once. Then lead variables $x_{i_1}, \ldots, x_{i_m}$ have been defined and the last system is a reduced echelon system.

1 Example (Elimination) Solve the system.

$$\begin{array}{rcrcrcrcr} w & + & 2x & - & y & + & z & = & 1, \\ w & + & 3x & - & y & + & 2z & = & 0, \\ & & x & & & + & z & = & -1. \end{array}$$

Solution

The answer using the natural variable list order w, x, y, z is the standard general solution

$$\begin{aligned} w &= 3 + t_1 + t_2, \\ x &= -1 - t_2, \\ y &= t_1, \\ z &= t_2, & \qquad -\infty < t_1, t_2 < \infty. \end{aligned}$$

Details appear in the next few slides.

Elimination Details

We will apply the three rules **swap**, **multiply** and **combination** for equivalent equations to obtain a frame which passes the Last Frame Test. Then we apply the Last Frame Algorithm to obtain the general solution of the system.

Let's mark processed equations with a box enclosing the lead variable (w is marked $\boxed{w}$).

$$\begin{array}{rcrcrcrcr} w & + & 2x & - & y & + & z & = & 1 \\ w & + & 3x & - & y & + & 2z & = & 0 \\ & & x & & & + & z & = & -1 \end{array}$$

1

$$\begin{array}{rcrcrcrcr} w & + & 2x & - & y & + & z & = & 1 \\ 0 & + & x & + & 0 & + & z & = & -1 \\ & & x & & & + & z & = & -1 \end{array}$$

2

$$\begin{array}{rcrcrcrcr} \boxed{w} & + & 2x & - & y & + & z & = & 1 \\ & & x & & & + & z & = & -1 \\ & & & & & & 0 & = & 0 \end{array}$$

3

$$\begin{array}{rcrcrcrcr} \boxed{w} & + & 0 & - & y & - & z & = & 3 \\ & & \boxed{x} & & & + & z & = & -1 \\ & & & & & & 0 & = & 0 \end{array}$$

4

Documentation of the steps

1 Original system. Identify the variable order as $\boldsymbol{w}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$.

2 Choose $\boldsymbol{w}$ as a lead variable. Eliminate $\boldsymbol{w}$ from equation 2 by using `combo(1,2,-1)`.

3 The $\boldsymbol{w}$-equation is processed. Let $\boldsymbol{x}$ be the next lead variable. Eliminate $\boldsymbol{x}$ from equation 3 using `combo(2,3,-1)`.

4 Eliminate $\boldsymbol{x}$ from equation 1 using `combo(2,1,-2)`. Mark the $\boldsymbol{x}$-equation as processed. **Last Frame Test** passed.

The four frames make the **frame sequence** which takes the original system into the last frame. Basic exposition rules apply:

1. Variables in an equation appear in variable list order.
2. Equations inherit variable list order from the lead variables.

Last Frame Algorithm

The last frame of the sequence, which must pass the Last Frame Test, is used to write out the general solution, as follows.

$$\begin{array}{rcrcrcr} \boxed{w} & = & 3 & + & y & + & z \\ \boxed{x} & = & -1 & - & z & & \\ y & = & t_1 & & & & \\ z & = & t_2 & & & & \end{array}$$

Solve for the lead variables $\boxed{w}$, $\boxed{x}$. Assign invented symbols t_1, t_2 to the free variables y, z.

$$\begin{array}{rcrcrcr} w & = & 3 & + & t_1 & + & t_2 \\ x & = & -1 & - & t_2 & & \\ y & = & t_1 & & & & \\ z & = & t_2 & & & & \end{array}$$

Back-substitute free variables into the lead variable equations to display the standard general solution.

Requirement: variables must be listed in variable list order $[w, x, y, z]$.