Strang: Chapter 4

Section 4.1. Exercises 5, 9, 11, 12, 16, 17, 19, 20, 21, 26

Section 4.2. Exercises 1, 2, 3, 11, 12, 17, 21, 27, 31

Section 4.3. Exercises 1, 6, 12, 17, 18, 21

Problem week9-1. Define a function T from $\mathcal{R}^n$ to $\mathcal{R}^m$ by the matrix multiply formula $T(\vec{x}) = A\vec{x}$. Prove that for all vectors $\vec{u}, \vec{v}$ and all constants c, (a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$, (b) $T(c\vec{u}) = cT(\vec{u})$. *Definition: T is called a* **linear transformation** *if T maps $\mathcal{R}^n$ into $\mathcal{R}^m$ and satisfies (a) and (b).*

Problem week9-2. Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^n$ that satisfies $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x}$. Prove that the $n \times n$ matrix A of T is orthogonal, that is, $A^T A = I$, which means the columns of A are **orthonormal**:

$$\mathbf{col}(A,i) \cdot \mathbf{col}(A,j) = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \mathbf{col}(A,i) \cdot \mathbf{col}(A,i) = 1.$$

Problem week9-3. Let T be a linear transformation given by $n \times n$ orthogonal matrix A. Then $\|T(\vec{x})\| = \|\vec{x}\|$ holds. Construct an example of such a matrix A for dimension $n = 3$, which corresponds to holding the z-axis fixed and rotating the xy-plane 45 degrees counter-clockwise. Draw a 3D-figure which shows the action of T on the unit cube $S = \{(x,y,z) \ : \ 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.

Section 4.4. Exercises 3, 6, 10, 13, 15, 18, 30, 32
4.4-18: It helps to find explicitly Q and R, which can be quickly checked in `Maple`.

Some Answers

4.1. Exercises 9, 12, 16, 21 have textbook answers.

4.1-5. The problem is about the equation nullspace = rowspace$^\perp$, valid for both matrix A and A^T. The **Fundamental Theorem of Linear Algebra**, Part II, can be summarized as the text

The nullspace is perpendicular to the rowspace.

The text is justified from equality of vectors: $Ax = 0$ is equivalent to scalar equations $\mathbf{row}(A,1) \cdot x = 0, \ldots, \mathbf{row}(A,m) \cdot x = 0$, which says that x is perpendicular to all rows of A, hence perpendicular to the rowspace of A.

(a) Apply the equation to A^T. Then nullspace(A^T) $\perp$ rowspace(A^T), equivalent to nullspace(A^T) $\perp$ colspace(A), implies any solution y to $A^T y = 0$ is perpendicular to any Ax. Since $b = Ax$, then $y \perp b$ or $y^T b = 0$.
(b) If $A^T y = (1,1,1)$ has a solution, then y is in rowspace(A). Then nullspace(A) $\perp$ rowspace(A) implies $y \cdot x = 0$ for all x in the nullspace of A.

4.1-9. Answers: colspace, perpendicular. See problem 4.1-5 above for nullspace $\perp$ rowspace, applied here for matrix A^T.
Prove $A^T Ax = 0$ **implies** $Ax = 0$.
Because $y = Ax$ is a linear combination of the columns of A, then y is in colspace(A) = rowspace(A^T). If $A^T Ax = 0$, then $A^T y = 0$, which implies y is in nullspace(A^T). Use nullspace(A^T) $\perp$ rowspace(A^T). Then nullspace(A) and rowspace(A^T) meet only in the vector $y = 0$, which says $y = Ax = 0$.
Prove $Ax = 0$ **implies** $A^T Ax = 0$.
First, assume $Ax = 0$. Multiply by A^T to get $A^T Ax = A^T 0$. The right side is the zero vector, which gives $A^T Ax = 0$.
See also problem 4.2-27, which repeats this same argument.

4.1-11.

For A: The nullspace is spanned by $(-2,1)$, the row space is spanned by $(1,2)$. The column space is the line through $(1,3)$ and $N(A^T)$ is the line through $(3,-1)$. In each case,

For B: The nullspace of B is is a line spanned by $(0,1)$, the row space is a line spanned by $(1,0)$. The column space and left nullspace are the same as for A. As in (a), the line pairs are perpendicular.

4.1-17. If S is the subspace of $\mathcal{R}^3$ containing only the zero vector, then $S^\perp$ is $\mathcal{R}^3$. If S is spanned by $(1,1,1)$, then $S^\perp$ is the plane spanned by any two independent vectors perpendicular to $(1,1,1)$. For example, the vectors $(1,-1,0)$ and $(1,0,-1)$. If S is spanned by $(2,0,0)$ and $(0,0,3)$, then $S^\perp$ is the line spanned by $(0,1,0)$, computed as the cross product of the two vectors, then scaled to be a unit vector.

4.1-26. $A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$

This example shows a matrix with perpendicular columns. Then $A^TA = 9I$ is diagonal: $(A^TA)_{ij} =$ (column i of A) $\cdot$ (column j of A). When the columns are unit vectors, then $A^TA = I$.

4.2. Exercises 1, 3, 11, 21, 31 have textbook answers.

4.2-2.

(a) The projection of $b = (\cos\theta, \sin\theta)$ onto $a = (1,0)$ is $p = (\cos\theta, 0)$.

(b) The projection of $b = (1,1)$ onto $a = (1,-1)$ is $p = (0,0)$ since $a^Tb = 0$.

4.2-12. $P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ = projection matrix onto the column space of A (the xy plane)

$P_2 = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.0 & 0.0 & 1 \end{pmatrix}$ = Projection matrix onto the second column space. Certainly $(P_2)^2 = P_2$.

4.2-17. If $P^2 = P$ then $(I-P)^2 = (I-P)(I-P) = I^2 - PI - IP + P^2 = I - P$. When P projects onto the column space, then $I - P$ projects onto the left nullspace.

4.2-27. If $A^TAx = 0$ then Ax is a vector in the nullspace of A^T. But Ax is a vector in the column space of A. To be in both of those perpendicular spaces, Ax must be zero. So A and A^TA have the same nullspace.

4.3. Exercises 1, 18, 21 have textbook answers.

4.3-6. $a = (1,1,1,1)$ and $b = (0,8,8,20)$ give $\hat{x} = \dfrac{a^Tb}{a^Ta} = 9$ and the projection is $\hat{x}a = p = (9,9,9,9)$. Then $e^Ta = (-9,-1,-1,11)^T(1,1,1,1) = 0$ and $\|e\| = \sqrt{204}$.

4.3-12.

(a) $a = (1,\ldots,1)$ has $a^Ta = m$, $a^Tb = b_1 + \cdots + b_m$. Therefore $\hat{x} = a^Tb/m$ is the mean of the b's

(b) $e = b - \hat{x}a$, $b = (1,2,b)$, $\|e\| = \sum_{i=1}^m (b_1 - \hat{x})^2 =$ variance

(c) $p = (3,3,3)$, $e = (-2,-1,3)$, $p^Te = 0$. $P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

4.3-17. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 21 \end{pmatrix}$. The solution $\vec{x} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$ comes from $\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$.

Problem week9-1. Write out both sides of identities (a) and (b), replacing $T(\vec{w})$ by matrix product $A\vec{w}$ for various choices of $\vec{w}$. Then compare sides to finish the proof.

Problem week9-2. Equation $\|T(\vec{x})\| = \|\vec{x}\|$ means lengths are preserved by T. It also means $\|A\vec{x}\| = \|\vec{x}\|$, which applied to $\vec{x} = \mathbf{col}(I,k)$ means $\mathbf{col}(A,k)$ has length equal to $\mathbf{col}(I,k)$ (=1). Write $\|\vec{w}\|^2 = \vec{w}\cdot\vec{w} = \vec{w}^T\vec{w}$ (the latter a matrix product). Then write out the equation $\|A\vec{x}\|^2 = \|\vec{x}\|^2$, to see what you get, for various choices of unit vectors $\vec{x}$.

Problem week9-3. The equations for such a transformation can be written as plane rotation equations in x, y plus the identity in z. They might look like $x' = x\cos\theta - y\sin\theta$, $y' =$ similar, $z' = z$. Choose θ then test it by seeing what happens to $x = 1$, $y = 0$, $z = 0$, the answer for which is a rotation of vector $(1, 0, 0)$. The answer for A is obtained by writing the scalar equations as a matrix equation $(x', y', z')^T = A(x, y, z)^T$.

4.4. Exercises 3, 6, 13, 15, 18, 30, 32 have textbook answers.

4.4-10. (a) If q_1, q_2, q_3 are orthonormal then the dot product of q_1 with $c_1q_1 + c_2q_2 + c_3q_3 = 0$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$. Independent q's. (b) $Qx = 0$ implies $x = 0$ implies $x = 0$.