

Math 2280 Extra Credit Problems
Chapter 9
S2017

Submitted work. Please submit one stapled package per chapter. Kindly label problems Extra Credit. Label each problem with its corresponding problem number, e.g., Xc1.2-4. Please attach this printed sheet to simplify your work.

Chapter 9: 9.1, 9.2, 9.3 – Periodic Functions and Fourier Series

Problem Xc9.0-1. (Trigonometric Identities and Integrals)

(a) Use the trigonometric identity $\cos(a + b) = \cos a \cos b - \sin a \sin b$ to derive the trigonometric identity

$$\cos mx \cos nx = \frac{1}{2} (\cos((m + n)x) + \cos((m - n)x)).$$

(b) Show the details for integrating $\cos mx \cos nx$ for nonnegative integers $m \neq n$ over $-\pi \leq x \leq \pi$.

(c) Derive the trigonometric identity $\cos^2 \theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta$ from the trigonometric identities $\cos(a + b) = \cos a \cos b - \sin a \sin b$ and $\cos^2 \theta + \sin^2 \theta = 1$.

(d) Integrate $\cos^2 nx$ for integers $n = 0, 1, 2, 3, \dots$ over $-\pi \leq x \leq \pi$. Explain geometrically why there are two different answers.

Problem Xc9.0-2. (Orthogonality)

Two vectors \vec{A}, \vec{B} are said to be **orthogonal** if their dot product is zero. For vectors of dimension n , this means $a_1 b_1 + a_2 b_2 + \dots + a_n b_n = 0$.

(a) The equation $\int_0^1 f(x)g(x)dx = 0$ can be viewed as *the Riemann sum is approximately zero*. Argue that this means $\vec{A} \cdot \vec{B} = 0$ to so many decimal places, where \vec{A} and \vec{B} are **vector samples** of f and g represented in the Riemann sum $h \sum_{j=1}^n f(jh)g(jh)$, $h = \frac{1}{n}$.

(b) Prove the orthogonality relation below from the standard one for the trigonometric system on $-\pi \leq x \leq \pi$, by a change of variables. The method leads to six orthogonality relations for the trigonometric system $\{\cos(m\pi x/T)\}_{m=0}^{\infty}$, $\{\sin(n\pi x/T)\}_{n=1}^{\infty}$ on $-T \leq u \leq T$.

$$\int_{-T}^T \cos(m\pi u/T) \cos(n\pi u/T) du = \begin{cases} 0 & m \neq n, \\ T & m = n. \end{cases}$$

Problem Xc9.1-1. (Periodic Functions)

(a) Find the period, amplitude and frequency of $\sin^2(4x)$.

(b) Let f be periodic of period 1 and on $0 \leq x \leq 1$ $f(x) = f_0(x)$, where $f_0(x) = 1$ on $0 \leq x < 1/2$, $f_0(x) = 0$ on $1/2 \leq x < 1$, $f_0(1) = 0$. Graph f on $-2 \leq x \leq 3$.

Problem Xc9.1-8. (Sums of Periodic Functions)

(a) Find the period of $\cos x + \cos 3x$.

(b) Find the period of $e^{2 \cos 2x}$.

Problem Xc9.1-8. (Periodic Functions)

Explain why $\cos x + \cos 3\pi x$ is not periodic.

Problem Notes. One explanation uses independence of functions. Another explanation analyzes the number of solutions of the equation $\cos x + \cos 2\pi x = 2$. Expected in this case is a graphic and a mathematical argument [use $-2 \leq f(x) \leq 2$].

Problem Xc9.1-18. (Change of Variables)

(a) Prove that $f(x)$ continuous and T -periodic implies $\int_0^T f(x)dx = \int_{nT}^{nT+T} f(u)du$.

(b) Assume $f(x)$ is 2π -periodic and continuous. Prove that $F(x) = \int_0^x f(u)du$ is 2π -periodic if and only if $\int_0^{2\pi} f(x)dx = 0$.

Problem Xc9.1-20. (Floor Function)

The greatest integer function or **staircase function** is represented using a library function **floor**(x), available in most computer mathematical workbenches, including **MATLAB** and **maple**. Don't confuse **floor** with **trunc** – they are different functions!

(a) Plot **floor**(x) and $x - \mathbf{floor}(x)$ in **MATLAB** or **maple** on $-3 \leq x \leq 5$. Programs **Excel** and **OpenOffice** can also be used to make the plot. The **floor** function in **Excel** is $x \rightarrow \mathbf{FLOOR}(x, 1)$.

(b) Argue from the graphic that $x - \mathbf{floor}(x)$ is periodic of period 1.

(c) Define $g(x, T) = x - T \mathbf{floor}((x + T/2)/T)$. Show mathematically that $g(x) = x$ on $|x| < T/2$. That the **triangular wave** g is T -periodic can be seen from its graphic.

(d) Plot the 2-periodic **triangular wave** defined by $f(x) = |x - 2 \mathbf{floor}((x + 1)/2)|$.

Problem Notes: The function f in (d) equals $|g(x, T)|$, where $T = 2$ and g is defined in (c) above. The function $a|g(x, T)|$ is called a **triangular wave** of height a and period T . It is the composition of $u \rightarrow a|u|$ and $x \rightarrow g(x, T)$.

Problem Xc9.2-5. (Fourier Series Partial Sum Plots)

(a) Plot on $-2\pi \leq x \leq 3\pi$ the partial sums $s_2(x), s_6(x), s_{12}(x), s_{20}(x)$ of the Fourier series for the sawtooth wave f constructed from $f_1(x) = |x|$ on $|x| \leq \pi$:

$$s_N(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^N \frac{1}{(2m+1)^2} \cos(2mx + x).$$

The four graphics show the convergence of the partial sums to the limiting Fourier series. This is an example of a *filmstrip* of 4 graphics.

(b) Explain what happens in the graphic of (a) at points of discontinuity of f' .

Problem Xc9.2-5a. (Fourier Series Partial Sum Plots)

(a) Plot on $-2\pi \leq x \leq 3\pi$ the partial sums $s_2(x), s_6(x), s_{12}(x), s_{20}(x)$ of the Fourier series for the sawtooth wave f constructed from $f_1(x) = (\pi - x)/2$ on $0 < x \leq 2\pi$:

$$s_N(x) = \sum_{n=1}^N \frac{1}{n} \sin(nx).$$

The four snapshots show the convergence of the partial sums to the limiting Fourier series.

(b) Explain what happens in the graphic of (a) at points of discontinuity of f .

(c) Illustrate Gibb's phenomenon. In particular, graph $|f(x) - s_N(x)|$ on $-2\pi \leq x \leq 3\pi$, for $N = 6, 12, 20$. Then estimate the jump at points of discontinuity of f' .

Answer: About 1.25.

Problem Xc9.2-8. (Fourier Series Computation and Graphics)

Let f be the 2π -periodic extension of the rectified cosine wave $f_0(x) = |\cos x|$ on $|x| \leq \pi$.

(a) Draw a graphic of $f(x)$ on $-4\pi \leq x \leq 5\pi$.

(b) Show the derivation details for the Fourier series $\frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{4m^2 - 1} \cos 2mx$.

(c) Plot the Fourier series on $-2\pi \leq x \leq 2\pi$. Explain why it differs from the plot of $f(x)$ on the same interval.

Problem Xc9.2-15. (Fourier Series Computation)

Show the derivation details for the Fourier coefficients of $f(x)$ constructed from $f_2(x) = e^{-|x|}$ on $|x| \leq \pi$. The Fourier series is

$$\frac{e^\pi - 1}{\pi e^\pi} + \frac{2}{\pi e^\pi} \sum_{m=1}^{\infty} \frac{e^\pi + (-1)^{m+1}}{m^2 + 1} \cos mx.$$

Problem Xc9.0-3. (Even and Odd Functions)

(a) Define *even function* and *odd function*. Such functions don't have to be continuous, but they must be defined for all x .

(b) Show the mathematical details in the derivation of the result $(Even)(Odd) = Odd$.

(c) Prove by a u -substitution that $\int_{-p}^p f(x)dx = 2 \int_0^p f(x)dx$ for an even continuous function f and $\int_{-p}^p g(x)dx = 0$ for an odd continuous function $g(x)$.

Problem Xc9.3-7. (Fourier Series Arbitrary Period)

(a) Define f to be the periodic extension of period 4 of the base function $f_0(x) = 1 - x$ on $0 \leq x \leq 2$, $f_0(x) = -1 - x$ on $-2 \leq x \leq 0$. Plot $f(x)$ on $-8 \leq x \leq 6$.

(b) Show the derivation details for the Fourier series of $f(x)$:

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{2n}.$$

Problem Xc9.3-32. (Failure of Term-by-Term Differentiation)

Show that the Fourier series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ of the sawtooth wave f cannot be differentiated term-by-term to obtain the Fourier series of f' .

Problem Xc9.3-34. (Term-by-Term Integration)

Integrate the Fourier series of the triangular wave f constructed from $f_0(x) = x$ on $|x| \leq 1$, in order to find the Fourier series of the parabolic wave g constructed from $g_0(x) = x^2$ on $|x| \leq 1$.

Chapter 9: 9.3, 9.4 – Fourier Series Methods

Problem Xc9.0-4. (Periodic Extensions)

Lemma 1. The function $\mathbf{tw}(x) = x - \text{floor}(x + 1/2)$ is a **triangular wave** of period 1 with shape x on $|x| < 1/2$.

Lemma 2. Given $f_0(x)$ defined on $|x| \leq T/2$, then $f(x) = f_0(T \mathbf{tw}(x/T))$ is the T -periodic extension of $f_0(x)$ from $|x| \leq T/2$ to $-\infty < x < \infty$.

Assume Lemmas 1 and 2 for this problem.

(a) Plot $f_1(x) = 3 \mathbf{tw}(x/3)$ on $-6 \leq x \leq 6$. Document its period on the graphic.

(b) Define $f_2(x) = |\cos(0.5\pi \mathbf{tw}(2x/\pi))|$. Make a plot on $-2\pi \leq x \leq 3\pi$. Document its period on the graphic.

Problem Xc9.0-5. (Even and Odd Periodic Extensions)

Definition. Define $\mathbf{signum}(x) = \begin{cases} \frac{x}{|x|} & x \neq 0, \\ 0 & x = 0. \end{cases}$

There is no agreement in literature how to define $\mathbf{signum}(0)$. Here, $\mathbf{signum}(x)$ takes on only the values 1, -1 and 0.

(a) Let $p = 2$ and define $g_1(x) = x^2$ on $0 \leq x \leq p$. Let $g_2(x) = \mathbf{signum}(x)g_1(|x|)$ be the odd extension of g_1 to $|x| \leq p$. Let $T = 2p$. Define $f_3(x) = g_2(T \mathbf{tw}(x/T))$ to be the odd extension of $g_2(x)$ from $|x| \leq p$ to $-\infty < x < \infty$. Plot f_3 on $|x| \leq 5$. This sequence of formulas works in general, for any p and any g_1 (no justification requested).

(b) Let $p = 2$ and define $h_1(x) = x^2$ on $0 \leq x \leq p$. Let $h_2(x) = h_1(|x|)$ be the even extension of h_1 to $|x| \leq p$. Let $T = 2p$. Define $h_3(x) = h_2(T \text{tw}(x/T))$ to be the even extension of $h_2(x)$ from $|x| \leq p$ to $-\infty < x < \infty$. Plot h_3 on $|x| \leq 5$. This sequence of formulas works in general, for any p and any h_1 (no justification requested).

Problem Xc9.0-6. (Dirichlet Kernel Identity)

Establish by trigonometric identity methods the formula [the right side is called **Dirichlet's Kernel**]

$$\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(nx + \frac{x}{2}\right)}{2 \sin\left(\frac{x}{2}\right)}.$$

Hint: Cross multiply by $2 \sin(x/2)$. Expand terms using a trigonometric identity, which produces a telescoping sum.

Problem Xc2.4-7. (Half-Range Expansions)

(a) Find a simple algebraic formula for the even π -periodic extension of $f_0(x) = \cos x$ on $0 \leq x \leq \pi/2$.

(b) Find the Fourier coefficients for the half-range expansion of $f_0(x) = \cos x$ on $0 \leq x \leq \pi/2$.

Problem Xc9.4-15. (Half-Range Sine Expansion)

Find the Fourier coefficients for the half-range sine series expansion of e^x on $0 \leq x \leq 1$.

Problem Xc9.4-6. (Complex Fourier Series)

Find the complex form of the Fourier series for $\sin 3x$ without evaluating any trigonometric integrals.

Hint: Use $\sin u = \frac{1}{2i}(e^{iu} - e^{-iu})$.

Problem Xc9.4-11. (Series Identities)

Let $x = 0$ in the complex Fourier series expansion of e^x in order to prove the formula

$$\frac{2\pi}{e^\pi - e^{-\pi}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + 1}.$$

Chapter 9: 9.5 – One Dimensional Heat Equation

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Problem Xc9.6-13. (Nonhomogeneous Heat Equation)

Consider the one-dimensional heat conduction problem

$$\begin{aligned} u_t &= u_{xx}, \quad 0 \leq x \leq \pi, \quad t > 0, \\ u(0, t) &= 100, \\ u(\pi, t) &= 50, \\ u(x, 0) &= f(x). \end{aligned}$$

Assume $f(x) = 33x$ on $0 < x \leq \pi/2$, $f(x) = 33\pi - 33x$ on $\pi/2 < x < \pi$. Find a solution formula for the temperature $u(x, t)$.

Problem Xc9.6-3. (Heat Conduction in an Insulated Bar)

Consider the one-dimensional heat conduction problem

$$\begin{aligned} u_t &= u_{xx}, \quad 0 \leq x \leq 1, \quad t > 0, \\ u_x(0, t) &= 0, \\ u_x(1, t) &= 0, \\ u(x, 0) &= \cos \pi x \end{aligned}$$

Find a solution formula for the temperature $u(x, t)$ at location x along the bar at time t . **Hint:** Don't integrate!

Remark. Asmar's matching problem 3.6-3 has a piecewise example, using $u(x, 0) = f(x)$. See the maple advice for problem 9.5-13, to handle that case.

Chapter 9: 9.6 – One Dimensional Wave Equation

Problem Xc9.6-1. (Wave Equation)

Derive the equation $u_{tt} = 10^5 u_{xx}$ for the vibrations of a stretched homogeneous string with linear density $\rho = 0.001$ kg/m and tension $\tau = 100$ N, with no forces other than the tension. State all assumptions used to obtain the model. Make the presentation brief, by referencing a textbook for derivation details and results.

Problem Xc9.6-9a. (Separation of Variables)

Solve $u_{tt} = u_{xx}$, $u(0, t) = u(1, t) = 0$, $u(x, 0) = x(1 - x)$, $u_t(x, 0) = \sin \pi x$, $t \geq 0$, $0 \leq x \leq 1$. The model is for a guitar string of unit length.

Problem Xc9.6-9b. (Filmstrip Plots)

Plot partial sums of the answer to the previous problem,

$$u(x, t) = \frac{1}{\pi} \sin(\pi x) \sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^3(2m+1)^3} \sin(2m\pi x + \pi x) \cos(2m\pi t + \pi t),$$

at $t = 0, 1, 2, 3$. Choose the number of series terms for the four graphics by making the first graphic match $x(1 - x)$ on $0 \leq x \leq 1$. This filmstrip has 4 frames, each frame corresponding to a time t . A frame has graph window $0 \leq x \leq 1$, $a \leq u \leq b$ (you must choose a, b).

Problem Xc9.6-9c. (Surface Plot)

Plot a specific partial sum of the answer

$$u(x, t) = \frac{1}{\pi} \sin(\pi x) \sin(\pi t) + \sum_{m=0}^{\infty} \frac{8}{\pi^3(2m+1)^3} \sin(2m\pi x + \pi x) \cos(2m\pi t + \pi t)$$

on the domain $0 \leq x \leq 1$, $0 \leq t \leq 4$. Use all features possible of the 3D graphics program in order to produce the best plot with fine accuracy, view and colors.

Problem Xc9.6-13. (Damped Vibrations of a String)

Solve the problem

$$\begin{aligned} u_{tt}(x, t) + u_t(x, t) &= u_{xx}(x, t), \\ u(0, t) &= 0, \\ u(\pi, t) &= 0, \\ u(x, 0) &= \sin x, \\ u_t(x, 0) &= 0. \end{aligned}$$

Problem Xc9.6-1. (Vibrating Finite String)

Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ with boundary and initial conditions $u(0, t) = u(L, t) = 0$, $u(x, 0) = \frac{1}{2} \sin \frac{\pi x}{L} + \frac{1}{4} \sin \frac{3\pi x}{L} + \frac{2}{5} \sin \frac{7\pi x}{L}$, $u_t(x, 0) = 0$, $0 < x < L$, $t \geq 0$. Use the series formula $u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{n\pi ct}{L}$.

References: Edwards-Penney section 9.6 (2280 textbook) and Asmar's text, *PDE and BVP*, section 1.2.

Problem Xc9.6-2. (Loudness)

The fraction of the loudness associated with the fundamental tone (b_1 -term in the series) is the quotient

$$F_1 = \frac{(n^2 b_n^2)|_{n=1}}{\sum_{k=1}^{\infty} k^2 b_k^2}$$

Find an approximation to the percentage $100F_1$.

References: ProbXc9.6-1. The discussion of music in E&P includes a derivation of the formula for the percentage loudness $100F_n$.

Chapter 9: 9.6 – d’Alembert’s Method

Problem Xc9.6-15. (d’Alembert’s Solution)

Consider the problem

$$\begin{aligned}u_{tt} &= u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0, \\u(0, t) &= 0, \\u(1, t) &= 0, \\u(x, 0) &= f(x), \\u_t(x, 0) &= 0.\end{aligned}$$

Assume $f(x) = 4x$ on $0 \leq x \leq 0.25$, $f(x) = 2 - 4x$ on $0.25 < x \leq 0.5$, $f(x) = 0$ on $0.5 < x \leq 1$.

(a) Find a solution formula for $u(x, t)$ using d’Alembert’s method.

(b) Plot a 3-frame filmstrip of the string shape at times $t = 0, 0.25, 0.5$.

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# EXAMPLE. Let f(x)=4x on [0,.25], f(x)=2-4x on [.25,.5], f(x)=0 otherwise
# Asmar 3.4-15, D’Alembert’s solution of the wave equation, f=pulses,g=0
pulse:=(x,a,b)->piecewise(x<a,0,x<b,1,0);
f:=x->4*x*pulse(x,0,1/4)+(2-4*x)*pulse(x,1/4,1/2);
#plot(f(x),x=0..1);
F:=x->piecewise(x<0,-f(-x),f(x)); # Odd extension of f(x)
plot(F(x),x=-1..1);
u:=(x,t)->(1/2)*(F(x+t)+F(x-t));
#plot(u(x,0.7),x=-2..2);
plots[animate]( plot, [u(x,t),x=-3..3], t=0..1.5, trace=0, frames=50 );
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Problem Xc9.6-18. (Energy Conservation and d’Alembert’s Solution)

Define

$$E(t) = \frac{1}{2} \int_0^L (u_t^2(x, t) + c^2 u_x^2(x, t)) dx.$$

Prove the energy conservation law, which says that the energy during free vibrations of a string is constant for all time.

Problem Notes. Show $dE/dt = 0$.

End of extra credit problems chapter 9.