MATH 2280-001, REVIEW PROBLEMS FOR THE FINAL EXAM

Note. The final exam is on Wednesday, December 16, from 8:00AM-10:00PM. It will be comprehensive, though with a greater emphasis on material since the last exam. Calculators or other electronic devices will not be allowed on the exam, however you will be supplied with the integral table. You will be allowed two sides of an 8.5 by 11 inch sheet of notes. I will be around most of the afternoon on Monday, and early afternoon on Tuesday.

Note. These are review problems and should not be considered a practice exam, though some of them come from an old final exam. I would also recommend looking over Exams 1 and 2, as well as the review problems for those.

1. Tank 1 contains 50 gal. of brine solution, and tank 2 contains 100 gal. of brine solution. Pure water flows into tank 1, the mixed solution flows from tank 1 into tank 2, and the solution drains from tank 2, all at a rate of 5 gal. a minute. Let $x(t)$ and $y(t)$ denote the amount of salt in tanks 1 and 2 respectively.

a. Derive the system of differential equations

$$x' = -0.1x$$
$$y' = 0.1x - 0.05y$$

b. Suppose that initially, $x(0) = 3$ and $y(0) = 0$. Find $x(t)$ and $y(t)$.

2. Find and classify the critical points (determine their stability as well as whether they are nodes, spirals, saddles etc...) of the following system:

$$x' = x^2 - 2xy + y^2 - 1$$
$$y' = 4 - y^2$$

3. Consider the differential equation:

$$y^{(4)} + 4y^{(3)} + 8y'' + 16y' + 16y = t + 2$$

a. Find the general solution of the corresponding homogeneous equation (hint: -2 is a root of the characteristic polynomial).

b. Find the general solution to the equation.

4. Consider the mass spring system as in figure 5.4.3 of the text (page 325) with $m_1 = 2$, $m_2 = 1/2$, $k_1 = 75$, and $k_2 = 25$.

a. Write down the mass-spring system $M\mathbf{x}'' + K\mathbf{x} = 0$, and find the fundamental frequencies.

b. Find the general solution of the system.

c. If an external force of $100\cos(10t)$ acts on m_2, and if $\mathbf{x}(0) = \mathbf{x}'(0) = 0$, find the motion $\mathbf{x}(t)$.

5. Let f be the function:

$$f(x) = \begin{cases} 1-x & \text{for} \ \ 0 \le x \le 1 \\ 0 & \text{for} \ \ 1 \le x \le 2 \end{cases}$$

a. Find the cosine series of f. Also, graph the cosine series of f as a function on $\mathbf{R}$ (include 3 periods).

b. Find the sine series of f. Also, graph the sine series of f as a function on $\mathbf{R}$ (include 3 periods).

6. Suppose that a laterally insulated rod of length 2ft. is made out of a material with thermal diffusivity $k = 0.1$, and suppose that at time $t = 0$, the initial temperature distribution is given by the function f from problem 5.

a. Suppose that the temperature at the ends of the rod are kept at 0 degrees. Write down the heat equation with appropriate boundary conditions. Find the solution $u(x,t)$.

b. Suppose instead, that that the endpoints are insulated. Find the corresponding temperature $u(x,t)$. What is the limiting temperature of the rod as t goes to infinity?

7. Suppose that a guitar string 2ft. long is under tension T, and has density ρ such that $a^2 = T/\rho = 6$. Let $u(x,t)$ denote the position of the string at x at time t. Assume that at $t = 0$ the string is plucked with initial position $u(x,0) = f(x)$ where f is defined below, and released with zero velocity.

$$f(x) = 0.2\sin(\frac{\pi}{2}x) - 0.1\sin(\pi x)$$

a. Write down the appropriate wave equation with boundary, and initial conditions.

b. Find the solution $u(x,t)$ (series form).

c. Find the D'Alembert form of the solution.