Solutions to Review Problems for Exam 2

1. $mx'' + cx' + kx = f(t)$

a. mx'' is mass · acceleration.
cx' denotes the frictional force on the mass
kx denotes the spring force on the mass
$f(t)$ is an external force acting on the mass

b. $0.25x'' + 4x = 0 \Rightarrow 0.25r^2 + 4 = 0$

$x(0) = 1$ $\quad r^2 + 16 = 0$
$x'(0) = 4$ $\quad r = \pm 4i$

Solution is of form $x = A\cos 4t + B\sin 4t$, $x' = -4A\sin 4t + 4B\cos 4t$

$x(0) = A = 1$
$x'(0) = +4B = 4$ $\Rightarrow A = 1,\ B = 1$

$x(t) = \cos 4t + \sin 4t$

c. The amplitude is $\sqrt{1^2+1^2} = \sqrt{2}$
The period is $\frac{2\pi}{4} = \frac{\pi}{2}$

$x(t) = A\cos(\omega t - \varphi)$ where A = amplitude $= \sqrt{2}$
ω = circular frequency $= 4$
$\tan\varphi = \frac{1}{1} = 1 \Rightarrow \varphi = \frac{\pi}{4}$

$x(t) = \sqrt{2}\cos(4t - \pi/4)$

d. Resonance will occur, when $f(t) = \cos\alpha t$, if $\alpha = \omega$ is the natural frequency of the system so $\boxed{\alpha = 4}$

e. $0.25x'' + x' + 4x = 0$

Characteristic roots: $0.25r^2 + r + 4 = 0$ or

$$r^2 + 4r + 16 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 4 \cdot 16}}{2} = \frac{-4 \pm \sqrt{48}\, i}{2}$$

Motion will be <u>underdamped</u>

f. From the undetermined coefficient method, x_p will have the form

$$x_p = A\cos t + B\sin t + Ce^{-t}\cos t + De^{-t}\sin t$$

for some constants A, B, C, D. Note, there is no "duplication"! None of these terms is a soln. of the homogeneous problem, by part e.

2. $$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a. $$\begin{vmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$-\lambda((1-\lambda)(-\lambda) - 0) - [(-1)(1-\lambda) - 0] = 0$$

$$\cancel{\lambda^3 + \lambda \neq 0} \quad \cancel{\text{or}} \quad \cancel{\lambda^3 - \lambda = 0 \Leftrightarrow \lambda(\lambda^2 + \lambda) \neq 0}$$

$$\Rightarrow \boxed{\lambda = 0, i, -i}$$

Factor out $1-\lambda$:

$$(1-\lambda)(\lambda^2+1)=0$$

$$\lambda = 1,\ +i,\ -i$$

Find eigenvectors:

$\underline{\lambda=1}$:
$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = x_2 = 0,\quad x_3 = x_3$ take $\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda = i}$ $(A-i)\vec{v}=0 \Rightarrow$
$$\left(\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 1 & 0 & 1-i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1-i & 0 \\ -1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 1-i & 0 \\ 0 & -i & 1-i & 0 \\ 0 & 1 & 1+i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1-i & 0 \\ 0 & 1 & 1+i & 0 \\ 0 & -i & 1-i & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1-i & 0 \\ 0 & 1 & 1+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\to$ $x_1 = -(1-i)\,x_3$
$x_2 = -(1+i)\,x_2$
$x_3 = x_3$

Set $x_3 = 1 \Rightarrow$ complex eigenvector:

$$\vec{v} = \begin{pmatrix} -1+i \\ -1-i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Complex Solution: expr in term of real + imag. parts

$$e^{it}\vec{v} = (\cos t + i\sin t)\left[\begin{pmatrix}-1\\-1\\1\end{pmatrix} + i\begin{pmatrix}1\\-1\\0\end{pmatrix}\right]$$

$$= \left[\cos t\begin{pmatrix}-1\\-1\\1\end{pmatrix} - \sin t\begin{pmatrix}1\\-1\\0\end{pmatrix}\right] + i\left[\cos t\begin{pmatrix}1\\-1\\0\end{pmatrix} + \sin t\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right]$$

Get soln's

$$\vec{x}_1(t) = e^{t}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\vec{x}_2(t) = \cos t\begin{pmatrix}-1\\-1\\1\end{pmatrix} - \sin t\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\vec{x}_3(t) = \cos t\begin{pmatrix}1\\-1\\0\end{pmatrix} + \sin t\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

General Solution:

$$\vec{X}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) \quad \text{or}$$

$$\boxed{\vec{X}(t) = c_1 e^{t}\begin{pmatrix}0\\0\\1\end{pmatrix} + c_2\cos t\begin{pmatrix}-1\\-1\\1\end{pmatrix} - c_2\sin t\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_3\cos t\begin{pmatrix}1\\-1\\0\end{pmatrix} + c_3\sin t\begin{pmatrix}-1\\-1\\1\end{pmatrix}}$$

3. In matrix form:

$$\vec{x}' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \vec{x}.$$

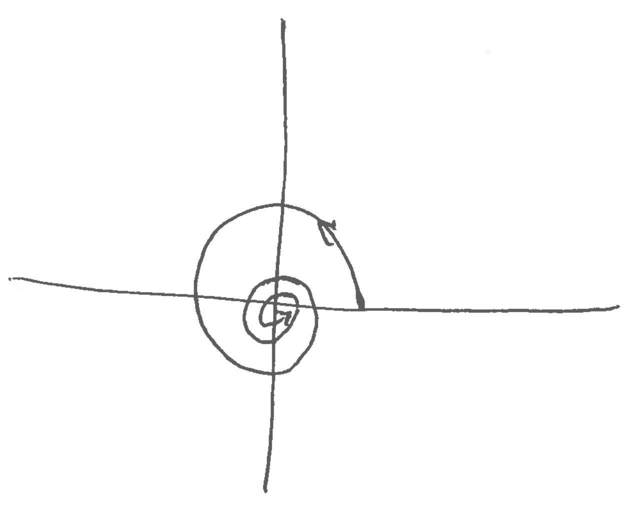
The eigenvalues are solns of $\begin{vmatrix} -1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (-1-\lambda)^2 + 1 = 0 \Rightarrow (\lambda+1)^2 = -1$$

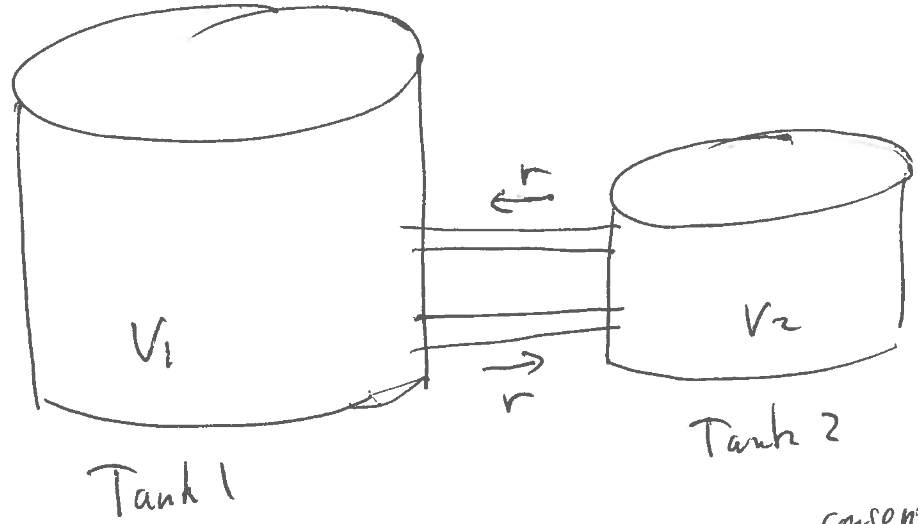
$$\Rightarrow \lambda + 1 = \pm i \Rightarrow \lambda = -1 \pm i$$

Thus $(0,0)$ will be an asymptotically stable spiral. To get direction, look at direction field at point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

The second component ~~pos~~ is positive so flows upwards at $(1,0)$ i.e. counterclockwise.



4.



$r = 10$

$V_1 = 100$

$V_2 = 50$

c_i = consentration in Tank i

$= \frac{x}{V_1}$ for tank 1

$\frac{y}{V_2}$ for tank 2

$x' = rc_2 - rc_1$

$y' = rc_1 - rc_2$

$\Rightarrow x' = -r\frac{x}{V_1} + r\frac{y}{V_2} \Rightarrow x' = -\frac{1}{10}x + \frac{1}{5}y$

$y' = r\frac{x}{V_1} - \frac{ry}{V_2} \qquad y' = \frac{1}{10}x - \frac{1}{5}y$

b. In matrix form $\vec{x}\,' = \begin{pmatrix} -1/10 & 1/5 \\ 1/10 & -1/5 \end{pmatrix} \vec{x}$.

eigenvalues: $\begin{vmatrix} -1/10 - \lambda & 1/5 \\ 1/10 & -1/5 - \lambda \end{vmatrix} = \left(\frac{1}{10} + \lambda\right)\left(\frac{1}{5} + \lambda\right) - \frac{1}{50} = \lambda^2 + \frac{3}{10}\lambda + \frac{1}{50} - \frac{1}{50}$

$\Rightarrow \lambda^2 + \frac{3}{10}\lambda = 0 \Rightarrow \lambda\left(\lambda + \frac{3}{10}\right) = 0 \qquad \lambda = 0,\ \lambda = -\frac{3}{10}$.

Eigenvectors: $\lambda = 0$: $\left(\begin{array}{cc|c} -1/10 & 1/5 & 0 \\ 1/10 & -1/5 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = 2x_2$: $\Rightarrow \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x_2 = x_2$ set $x_2 = 1$

$\lambda = -\frac{3}{10}$: $\left(\begin{array}{cc|c} -1/10 + 3/10 & 1/5 & 0 \\ 1/10 & -1/5 + 3/10 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} \frac{1}{5} & \frac{1}{5} & 0 \\ 1/10 & 1/10 & 0 \end{array}\right)$

$\to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to$ $x_1 = -x_2$. Set $x_2 = 1$

$x_1 = -1$

$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

General soln.: $\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} v_2$

or $\boxed{\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-\frac{3}{10} t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

c. $\vec{x}(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ Solve $\vec{x}(0) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$

or $\left(\begin{array}{cc|c} 2 & -1 & x_0 \\ 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & x_0 \end{array}\right)$

$\to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & x_0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & -\frac{x_0}{3} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & x_0/3 \\ 0 & 1 & -x_0/3 \end{array}\right)$

$c_1 = x_0/3$, $c_2 = -x_0/3$

$\vec{x}(t) = \frac{x_0}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{x_0}{3} e^{-3/10\, t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\lim_{t \to \infty} = \frac{x_0}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

so in limit: $\boxed{\begin{aligned} x &= \tfrac{2}{3} x_0 \\ y &= \tfrac{1}{3} x_0 \end{aligned}}$