Sample Quiz 11

Background. Switches and Impulses

Laplace’s method solves differential equations. It is the preferred method for solving equations containing switches or impulses.

Unit Step Define \( u(t-a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases} \). It is a switch, turned on at \( t = a \).

Ramp Define \( \text{ramp}(t-a) = (t-a)u(t-a) = \begin{cases} t-a & t \geq a, \\ 0 & t < a. \end{cases} \), whose graph shape is a continuous ramp at 45-degree incline starting at \( t = a \).

Unit Pulse Define \( \text{pulse}(t,a,b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} = u(t-a) - u(t-b) \). The switch is ON at time \( t = a \) and then OFF at time \( t = b \).

Impulse of a Force

Define the \text{impulse} of an applied force \( F(t) \) on time interval \( a \leq t \leq b \) by the equation

\[
\text{Impulse of } F = \int_a^b F(t)dt = \left( \int_a^b F(t)dt \right) (b-a) = \text{Average Force} \times \text{Duration Time}.
\]

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the \text{impulse} of the force is deemed important, and not its magnitude nor duration.

Define the \textbf{Dirac Unit Impulse} by the equation \( \delta(t-a) = \frac{du}{dt}(t-a) \), where \( u(t-a) \) is the unit step. Symbol \( \delta \) makes sense only under an integral sign, and the integral in question must be a generalized Riemann integral (definition pending), with new evaluation rules. Symbol \( \delta \) is an abbreviation like \textit{etc} or \textit{e.g.}, because it abbreviates a paragraph of descriptive text.

- Symbol \( M\delta(t-a) \) represents an ideal impulse of magnitude \( M \) at time \( t = a \). Value \( M \) is the change in momentum, but \( M\delta(t-a) \) contains no detail about the applied force or the duration. A common force approximation for a hammer hit of very small duration \( 2h \) and impulse \( M \) is Dirac’s approximation

\[
F_h(t) = \frac{M}{2h} \text{pulse}(t, a-h, a+h).
\]

- Symbol \( \delta \) is not manipulated as an ordinary function. It is a special modeling tool with rules for application and rules for algebraic manipulation.

\textbf{THEOREM} (Second Shifting Theorem). Let \( f(t) \) and \( g(t) \) be piecewise continuous and of exponential order. Then for \( a \geq 0 \),

\[
e^{-as} \mathcal{L}(f(t)) = \mathcal{L}\left( f(t)u(t)|_{t=t-a} \right),
\]

\[
\mathcal{L}(g(t)u(t-a)) = e^{-as} \mathcal{L}\left( g(t)|_{t=t+a} \right).
\]
Problem 1. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for the unit step, ramp and pulse, in these special cases:

\[ L(10u(t - \pi)) \] (2) \[ L(\text{ramp}(t - 2\pi)) \], (3) \[ L(10 \text{pulse}(t, 3, 5)) \].

(b) Backward table. Find \( f(t) \) in the following special cases.

\[ L(f) = \frac{5e^{-3s}}{s} \] (2) \[ L(f) = \frac{e^{-4s}}{s^2} \] (3) \[ L(f) = \frac{5}{s} \left( e^{-2s} - e^{-3s} \right) \].
Problem 2. Solve the following Dirac impulse problems.

(c) Dirac Impulse and the Second Shifting theorem. Solve the following forward table problems.

\[
\begin{align*}
(1) \ & \mathcal{L}(10\delta(t - \pi)), \quad (2) \ & \mathcal{L}(5\delta(t - 1) + 10\delta(t - 2) + 15\delta(t - 3)), \quad (2) \ & \mathcal{L}((t - \pi)\delta(t - \pi)).
\end{align*}
\]

The sum of Dirac impulses in (2) is called an **impulse train**.
Solutions

Solution (a). The forward second shifting theorem applies.

1. \( \mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) \) where \( g(t) = 10 \) and \( a = \pi \). Then \( \mathcal{L}(10u(t - \pi)) = \mathcal{L}(g(t)u(t - a)) = e^{-as} \mathcal{L}\left(g(t)\big|_{t=t+a}\right) = e^{-\pi s} \mathcal{L}\left(10\big|_{t=t+\pi}\right) = \frac{10}{s}e^{-\pi s} \).

2. \( \mathcal{L}(\text{ramp}(t - 2\pi)) = \mathcal{L}((t - 2\pi)u(t - 2\pi)) = \mathcal{L}\left(tu(t)\big|_{t=-2\pi}\right) = e^{-2\pi s} \mathcal{L}(t) = \frac{1}{s}e^{-2\pi s} \).

3. \( \mathcal{L}(10 \text{pulse}(t, 3, 5)) = 10 \mathcal{L}(u(t - 3) - u(t - 5)) = \frac{10}{s}(e^{3s} - e^{-5s}) \).

Solution (b). Presence of an exponential \( e^{-as} \) signals step \( u(t - a) \) in the answer, the main tool being the backward second shifting theorem.

1. \( \mathcal{L}(f) = \frac{5e^{-3s}}{s} = e^{-3s} \mathcal{L}(5) = \mathcal{L}(5u(t)\big|_{t=t+3}) = \mathcal{L}(5u(t - 3)) \). Lerch implies \( f = 5u(t - 3) \).

2. \( \mathcal{L}(f) = \frac{e^{-4s}}{s} = \frac{e^{-as}}{s} \) where \( a = 4 \). Then \( \mathcal{L}(f) = \frac{e^{-as}}{s} = \mathcal{L}(tu(t)\big|_{t=t-a}) = \mathcal{L}((t - 4)u(t - 4)) = \mathcal{L}(\text{ramp}(t - 4)) \). Lerch implies \( f = \text{ramp}(t - 4) \).

3. \( \mathcal{L}(f) = e^{-2s} \frac{5}{s} - e^{-3s} \frac{5}{s} = \mathcal{L}(5u(t - 2)) - \mathcal{L}(5u(t - 3)) = \mathcal{L}(5 \text{pulse}(t, 2, 3)) \). Lerch implies \( f = 5 \text{pulse}(t, 2, 3) \).

Solution (c). The main result for Dirac unit impulse \( \delta \) is the equation

\[
\int_0^\infty g(t)\delta(t - a)dt = g(a),
\]

valid for \( g(t) \) continuous on \( 0 \leq t < \infty \). When \( g(t) = e^{-st} \), then the equation implies the Laplace formula \( \mathcal{L}(\delta(t - a)) = e^{-as} \).

1. \( \mathcal{L}(10\delta(t - \pi)) = 10e^{-\pi s} \), by the displayed equation with \( g(t) = 10e^{-st} \), or by using linearity and the formula \( \mathcal{L}(\delta(t - a)) = e^{-as} \).

2. \( \mathcal{L}(5\delta(t - 1) + 10\delta(t - 2) + 15\delta(t - 3)) = 5 \mathcal{L}(\delta(t - 1)) + 10 \mathcal{L}(\delta(t - 2)) + 15 \mathcal{L}(\delta(t - 3)) = 5e^{-s} + 10e^{-2s} + 15e^{-3s} \).

3. \( \mathcal{L}((t - \pi)\delta(t - 2\pi)) = \int_0^\infty (t - \pi)e^{st}\delta(t - 2\pi)dt = (t - \pi)e^{-2\pi s} \), using \( g(t) = (t - \pi)e^{-st} \) and \( a = 2\pi \) in the equation.
Let

**Problem 3. Experiment to Find the Transfer Function \( h(t) \)**

Consider a second order problem

\[
ax''(t) + bx'(t) + cx(t) = f(t)
\]

which by Laplace theory has a particular solution defined as the convolution of the transfer function \( h(t) \) with the input \( f(t) \),

\[
x_p(t) = \int_0^t f(w)h(t-w)dw.
\]

Examined in this problem is another way to find \( h(t) \), which is the system response to a Dirac unit impulse with zero data. Then \( h(t) \) is the solution of

\[
ah''(t) + bh'(t) + ch(t) = \delta(t), \quad h(0) = h'(0) = 0.
\]

**The Problem.** Assume \( a, b, c \) are constants and define \( g(t) = \int_0^t h(w)dw \).

(a) Show that \( g(0) = g'(0) = 0 \), which means \( g \) has zero data.

(b) Let \( u(t) \) be the unit step. Argue that \( g \) is the solution of

\[
ag''(t) + bg'(t) + cg(t) = u(t), \quad g(0) = g'(0) = 0.
\]

The fundamental theorem of calculus says that \( h(t) = g'(t) \). Therefore, to compute the transfer function \( h(t) \), find the response \( g(t) \) to the unit step with zero data, followed by computing the derivative \( g'(t) \), which equals \( h(t) \).

The experimental impact is important. Turning on a switch creates a unit step, generally easier than designing a hammer hit.

(c) Illustrate the method for finding the transfer function \( h(t) \) in the special case

\[
x''(t) + 2x'(t) + 5x(t) = f(t).
\]

**Solutions**

(a) \( g(0) = \int_0^0 h(w)dw = 0, \ g'(0) = h'(0) = 0 \).

(b) Let \( u(t) \) be the unit step. Initial data was decided in part (a). The Laplace applied to

\[
a g''(t) + b g'(t) + c g(t) = u(t) \quad \text{gives} \quad (as^2 + bs + c) \mathcal{L}(g) = \mathcal{L}(u(t)).
\]

Then \( \mathcal{L}(g) = \mathcal{L}(h(t)) \mathcal{L}(u(t)) = \mathcal{L}(h(t)) \frac{1}{2} \mathcal{L} \left( \int_0^t h(r)dr \right) \) by the integral theorem. Lerch’s theorem then says \( g(t) = \int_0^t h(r)dr \).

(c) For equation \( x''(t) + 2x'(t) + 5x(t) = f(t) \) we replace \( x(t) \) by \( g(t) \) and \( f(t) \) by the unit step \( u(t) \), then solve \( g''(t) + 2g'(t) + 5g(t) = u(t) \), obtaining \( \mathcal{L}(g) = \frac{1}{s^2+2s+5} = \mathcal{L}(\frac{1}{5} - \frac{1}{10}e^{-t}(2\cos(2t) + \sin(2t))) \). Then \( g(t) = \frac{1}{5} - \frac{1}{10}e^{-t}(2 \cos(2t) + \sin(2t)) \) and \( h(t) = g'(t) = \frac{1}{2}e^{-t} \sin(2t) \).