Quiz 4, Problem 1. Picard-Lindelöf Theorem and Spring-Mass Models

Picard-Lindelöf Theorem. Let $\vec{f}(x, \vec{y})$ be defined for $|x-x_0| \leq h$, $||\vec{y}-\vec{y}_0|| \leq k$, with \vec{f} and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant H, 0 < H < h, the problem

$$\begin{cases} \vec{y}'(x) = \vec{f}(x, \vec{y}(x)), & |x - x_0| < H, \\ \vec{y}(x_0) = \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x-x_0| < H$.





Emile Picard

The Problem. The second order problem

$$\begin{cases} u'' + 2u' + 17u = 100, \\ u(0) = 1, \\ u'(0) = -1 \end{cases}$$
 (1)

is a spring-mass model with damping and constant external force. The variables are time x in seconds and elongation u(x) in meters, measured from equilibrium. Coefficients in the equation represent mass m=1 kg, a viscous damping constant c=2, Hooke's constant k=17 and external force F(x)=100.

Convert the scalar initial value problem into a vector problem, to which Picard's vector theorem applies, by supplying details for the parts below.

(a) The conversion uses the **position-velocity substitution** $y_1 = u(x), y_2 = u'(x)$, where y_1, y_2 are the invented components of vector \vec{y} . Then the initial data u(0) = 1, u'(0) = -1 converts to the vector initial data

$$\vec{y}(0) = \left(\begin{array}{c} 1\\ -1 \end{array}\right).$$

(b) Differentiate the equations $y_1 = u(x), y_2 = u'(x)$ in order to find the scalar system of two differential equations, known as a **dynamical system**:

$$y_1' = y_2, \quad y_2' = -17y_1 - 2y_2 + 100.$$

(c) The derivative of vector function $\vec{y}(x)$ is written $\vec{y}'(x)$ or $\frac{d\vec{y}}{dx}(x)$. It is obtained by componentwise differentiation: $\vec{y}'(x) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$. The vector differential equation model of scalar system (1) is

$$\begin{cases}
\vec{y}'(x) = \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \\
\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{cases} (2)$$

(d) System (2) fits the hypothesis of Picard's theorem, using symbols

$$\vec{f}(x, \vec{y}) = \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The components of vector function \vec{f} are continuously differentiable in variables x, y_1, y_2 , therefore \vec{f} and $\frac{\partial \vec{f}}{\partial \vec{v}}$ are continuous.

Quiz4 Problem 2. The velocity of a crossbow bolt launched upward from the ground was determined from a video and a speed gun to complete the following table.

Time t in seconds	Velocity $v(t)$ in ft/sec	Location
0.000	60	Ground
1.7	0	Maximum
3.5	-52	Near Ground Impact



(a) The bolt velocity can be approximated by a quadratic polynomial

$$v(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients a, b, c. Then solve for the coefficients.

- (b) Assume a linear drag model $v' = -32 \rho v$. Substitute the polynomial answer of (a) into this differential equation, then substitute t = 0 and solve for $\rho \approx 0.11$.
- (c) Solve the model $w' = -32 \rho w$, w(0) = 60 with $\rho = 0.11$.
- (d) The error between v(t) and w(t) can be measured. Is the drag coefficient value $\rho = 0.11$ reasonable?

References. Edwards-Penney sections 2.3, 3.1, 3.2. Course documents on Linear algebraic equations and Newton kinematics.

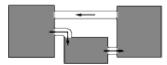
Quiz4 Extra Credit Problem 3. Consider the system of differential equations

$$x'_{1} = -\frac{1}{5}x_{1} + \frac{1}{7}x_{3},$$

$$x'_{2} = \frac{1}{5}x_{1} - \frac{1}{3}x_{2},$$

$$x'_{3} = \frac{1}{3}x_{2} - \frac{1}{7}x_{3},$$

for the amounts x_1, x_2, x_3 of salt in recirculating brine tanks, as in the figure:



Recirculating Brine Tanks A, B, C

The volumes are 50, 30, 70 for A, B, C, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x'_1 = x'_2 = x'_3 = 0$ and then solving for the symbols x_1, x_2, x_3 .

- (a) Solve the corresponding linear system of algebraic equations for answers x_1, x_2, x_3 .
- (b) The total amount of salt is uniformly distributed in the tanks in ratio 5:3:7. Explain this mathematically from the answer in (a).

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5. Course documents on Linear algebraic equations and Systems and Brine Tanks.