Quiz 8

Quiz 8, Problem 1. Solving Higher Order Constant-Coefficient Equations

The Algorithm applies to constant-coefficient homogeneous linear differential equations of order \( N \), for example equations like

\[
y'' + 16y = 0, \quad y''' + 4y'' = 0, \quad \frac{d^5y}{dx^5} + 2y''' + y'' = 0.
\]

1. Find the \( N \)th degree characteristic equation by Euler’s substitution \( y = e^{rx} \). For instance, \( y'' + 16y = 0 \) has characteristic equation \( r^2 + 16 = 0 \), a polynomial equation of degree \( N = 2 \).

2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the \( N \) roots according to multiplicity.

3. Construct \( N \) distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients \( c_1, c_2, c_3, \ldots \).

The solution space is then \( S = \text{span}(\text{the } N \text{ Euler solution atoms}) \).

Examples: Constructing Euler Solution Atoms from roots.

Three roots 0, 0, 0 produce three atoms \( e^{0x}, xe^{0x}, x^2e^{0x} \) or 1, \( x, x^2 \).

Three roots 0, 0, 2 produce three atoms \( e^{0x}, xe^{0x}, e^{2x} \).

Two complex conjugate roots \( 2 \pm 3i \) produce two atoms \( e^{2x}\cos(3x), e^{2x}\sin(3x) \).

Explained. The Euler substitution \( y = e^{rx} \) produces a solution of the differential equation when \( r \) is a complex root of the characteristic equation. Complex exponentials are not used directly. Ever. They are replaced by sines and cosines times real exponentials, which are Euler solution atoms. Euler’s formula \( e^{i\theta} = \cos \theta + i\sin \theta \) implies \( e^{2x}\cos(3x) = e^{2x}e^{3i\theta} + e^{2x}e^{-3i\theta} \), which is a linear combination of complex exponentials, solutions of the differential equation because of Euler’s substitution. Superposition implies \( e^{2x}\cos(3x) \) is a solution. Similar for \( e^{2x}\sin(3x) \).

The independent pair \( e^{2x}\cos(3x), e^{2x}\sin(3x) \) replaces both \( e^{(2+3i)x} \) and \( e^{(2-3i)x} \).

Four complex conjugate roots listed according to multiplicity as \( 2 \pm 3i, 2 \pm 3i \) produce four atoms \( e^{2x}\cos(3x), e^{2x}\sin(3x), x e^{2x}\cos(3x), x e^{2x}\sin(3x) \).

Seven roots 1, 1, 3, 3, 3, \pm 3i produce seven atoms \( e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x) \).

Two conjugate complex roots \( a \pm bi \) arising from roots of \( (r-a)^2 + b^2 = 0 \) produce two atoms \( e^{ax}\cos(bx), e^{ax}\sin(bx) \).

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) \( y'' + 4y' = 0 \)
(b) \( y'' + 4y = 0 \)
(c) \( y''' + 4y' = 0 \)
(d) \( y'' + 4y = 0, \ y(0) = 1, \ y'(0) = 2 \)
(e) \( y''' + 81y'' = 0, \ y(0) = y'(0) = 0, \ y''(0) = y'''(0) = 1 \)
(f) The characteristic equation is \( (r+1)^2(r^2-1) = 0 \).
(g) The characteristic equation is \( (r-1)^2(r^2-1)^2(r+1)^2 + 9 = 0 \).
(h) The characteristic equation roots, listed according to multiplicity, are 0, 0, -1, 2, 2, 3+4i, 3–4i, 3+4i, 3–4i.
Quiz 8, Problem 2. Laplace Theory

Laplace theory implements the *method of quadrature* for higher order differential equations, linear systems of differential equations, and certain partial differential equations.

Laplace’s method solves differential equations.

**The Problem.** Solve by table methods or Laplace’s method.

(a) Forward table. Find \( L(f(t)) \) for \( f(t) = 3(t + 1)^2e^{2t} + 2e^t \sin(3t) \).

(b) Backward table. Find \( f(t) \) for

\[
L(f(t)) = \frac{4s}{s^2 + 4} + \frac{s - 1}{s^2 - 2s + 5}.
\]

(c) Solve the initial value problem \( x''(t) + 2x'(t) + 5x(t) = e^t, \ x(0) = 0, \ x'(0) = 1. \)