Background. Switches and Impulses

Laplace’s method solves differential equations. It is the premier method for solving equations containing switches or impulses.

Unit Step Define \( u(t - a) = \begin{cases} 1 & t \geq a, \\ 0 & t < a. \end{cases} \), it is a switch, turned on at \( t = a \).

Ramp Define \( \text{ramp}(t - a) = (t - a)u(t - a) = \begin{cases} t - a & t \geq a, \\ 0 & t < a. \end{cases} \), whose graph shape is a continuous ramp at 45-degree incline starting at \( t = a \).

Unit Pulse Define \( \text{pulse}(t, a, b) = \begin{cases} 1 & a \leq t < b, \\ 0 & \text{otherwise} \end{cases} = u(t - a) - u(t - b). \) The switch is ON at time \( t = a \) and then OFF at time \( t = b \).

Impulse of a Force

Define the impulse of an applied force \( F(t) \) on time interval \( a \leq t \leq b \) by the equation

\[
\text{Impulse of } F = \int_a^b F(t)dt = \left( \int_a^b F(t)dt \right) / (b - a) = \text{Average Force \times Duration Time.}
\]

Dirac Unit Impulse

A Dirac impulse acts like a hammer hit, a brief injection of energy into a system. It is a special idealization of a real hammer hit, in which only the impulse of the force is deemed important, and not its magnitude nor duration.

Define the Dirac Unit Impulse by the equation \( \delta(t - a) = \frac{du}{dt}(t - a) \), where \( u(t - a) \) is the unit step. Symbol \( \delta \) makes sense only under an integral sign, and the integral in question must be a generalized Riemann-Stieltjes integral (definition pending), with new evaluation rules. Symbol \( \delta \) is an abbreviation like \( \text{etc} \) or \( \text{e.g.} \), because it abbreviates a paragraph of descriptive text.

- Symbol \( M\delta(t - a) \) represents an ideal impulse of magnitude \( M \) at time \( t = a \). Value \( M \) is the change in momentum, but \( M\delta(t - a) \) contains no detail about the applied force or the duration. A common force approximation for a hammer hit of very small duration \( 2h \) and impulse \( M \) is Dirac’s approximation

\[
F_h(t) = \frac{M}{2h} \text{pulse}(t, a - h, a + h).
\]

- The fundamental equation is \( \int_{-\infty}^{\infty} F(x)\delta(x - a)dx = F(a). \) Symbol \( \delta(t - a) \) is not manipulated as an ordinary function, but regarded as \( du(t - a)/dt \) in a Riemann-Stieltjes integral.

THEOREM (Second Shifting Theorem). Let \( f(t) \) and \( g(t) \) be piecewise continuous and of exponential order. Then for \( a \geq 0 \),

**Forward table**

\[
\mathcal{L}(f(t-a)u(t-a)) = e^{-as}\mathcal{L}(f(t)) \quad \mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t=t+a}\right)
\]

**Backward table**

\[
e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a)) \quad e^{-as}\mathcal{L}(f(t)) = \mathcal{L}\left(f(t)u(t)|_{t=t-a}\right)
\]

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Problem 1. Solve the following by Laplace methods.

(a) Forward table. Compute the Laplace integral for terms involving the unit step, ramp and pulse, in these special cases:

\[ \mathcal{L}(t - 1)u(t - 1) \quad (2) \quad \mathcal{L}(e^t \text{ramp}(t - 2)) \quad (3) \quad \mathcal{L}(5 \text{pulse}(t, 2, 4)). \]

(b) Backward table. Find \( f(t) \) in the following special cases.

\[ (1) \quad \mathcal{L}(f) = \frac{e^{-2s}}{s} \quad (2) \quad \mathcal{L}(f) = \frac{e^{-s}}{(s + 1)^2} \quad (3) \quad \mathcal{L}(f) = e^{-s} \frac{3}{s} - e^{-2s} \frac{3}{s}. \]
Problem 2. Solve the following Dirac impulse problem.

(c) Dirac Impulse and the Second Shifting theorem. Solve the following forward table problems.

\begin{align*}
(1) \ L(2\delta(t - 5)), \quad (2) \ L(2\delta(t - 1) + 5\delta(t - 3)), \quad (3) \ L(e^t\delta(t - 2)).
\end{align*}

The sum of Dirac impulses in (2) is called an impulse train. The numbers 2 and 5 represent the applied impulse at times 1 and 3, respectively.
Reference: The Riemann-Stieltjes Integral

Definition
The Riemann-Stieltjes integral of a real-valued function $f$ of a real variable with respect to a real monotone non-decreasing function $g$ is denoted by

$$\int_a^b f(x) \, dg(x)$$

and defined to be the limit, as the mesh of the partition

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

of the interval $[a, b]$ approaches zero, of the approximating Riemann-Stieltjes sum

$$S(P, f, g) = \sum_{i=0}^{n-1} f(c_i)(g(x_{i+1}) - g(x_i))$$

where $c_i$ is in the $i$-th subinterval $[x_i, x_{i+1}]$. The two functions $f$ and $g$ are respectively called the integrand and the integrator.

The limit is a number $A$, the value of the Riemann-Stieltjes integral. The meaning of the limit: Given $\varepsilon > 0$, then there exists $\delta > 0$ such that for every partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ with $\text{mesh}(P) = \max_{0 \leq i < n} (x_{i+1} - x_i) < \delta$, and for every choice of points $c_i$ in $[x_i, x_{i+1}]$,

$$|S(P, f, g) - A| < \varepsilon.$$