Picard–Lindelöf Theorem: The Vector Case

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Picard-Lindelöf Theorem. Let \( \vec{f}(x, \vec{y}) \) be defined for \( |x - x_0| \leq h \), \( \|\vec{y} - \vec{y}_0\| \leq k \), with \( \vec{f} \) and \( \frac{\partial \vec{f}}{\partial \vec{y}} \) continuous. Then for some constant \( H, 0 < H < h \), the problem

\[
\begin{cases}
    \vec{y}'(x) = \vec{f}(x, \vec{y}(x)), & |x - x_0| < H, \\
    \vec{y}(x_0) = \vec{y}_0
\end{cases}
\]

has a unique solution \( \vec{y}(x) \) defined on the smaller interval \( |x - x_0| < H \).
Conversion of Second Order Scalar to a First Order System

Example. Transform the spring-mass system into a first order system in vector form.

\[ y'' + 3y' + 2y = g(x), \quad y(0) = y_0, \quad y'(0) = y_1. \]

Let \( \vec{u}(x) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} \). Then \( u_1 = y(x), \ u_2 = y'(x) \) and

\[ \vec{u}'(x) = \begin{pmatrix} y'(x) \\ y''(x) \end{pmatrix} = \begin{pmatrix} y'(x) \\ g(x) - 3y'(x) - 2y(x) \end{pmatrix}, \]

because of the differential equation \( y'' + 3y' + 2y = g(x) \). Use \( y(x) = u_1, \ y'(x) = u_2 \) to write

\[ \vec{u}'(x) = \begin{pmatrix} y'(x) \\ g(x) - 3y'(x) - 2y(x) \end{pmatrix} = \begin{pmatrix} u_2 \\ g(x) - 3u_2 - 2u_1 \end{pmatrix}. \]

Define \( \vec{f}(x, \vec{u}) = \begin{pmatrix} u_2 \\ g(x) - 3u_2 - 2u_1 \end{pmatrix}. \)

Then \( \vec{u}' = \vec{f}(x, \vec{u}) \) is the vector form of the spring-mass system.

The initial condition is \( \vec{u}(0) = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}. \)
Definitions

**Definition.** A vector function \( \vec{f}(x, \vec{u}) \) is said to be continuous on a set \( |x - x_0| < h, \|\vec{u} - \vec{u}_0\| < H \) provided for each \( (x_1, \vec{u}_1) \) in the set, we have

\[
\lim_{x \to x_1, \vec{u} \to \vec{u}_1} \vec{f}(x, \vec{u}) = \vec{f}(x_1, \vec{u}_1).
\]

**Definition.** Symbol \( \partial \vec{f}(x, \vec{u}) / \partial \vec{u} \) is the Jacobian matrix of partial derivatives of vector \( \vec{f} \) with respect to the components of vector \( \vec{u} \). If \( \vec{f} = \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) \) and \( \vec{u} = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) \), then

\[
\frac{\partial \vec{f}(x, \vec{u})}{\partial \vec{u}} = \left( \begin{array}{cc} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{array} \right).
\]
Jacobians and Continuity

A Jacobian matrix is said to be continuous provided all of its entries are continuous. This implies:

**Theorem.** A Jacobian matrix of $\vec{f}$ is continuous in variables $x, \vec{u}$ provided the partial derivatives $\partial \vec{f}(x, \vec{u})/\partial u_j, j = 1, \ldots, n$, are continuous in variables $x, \vec{u}$.

**Example.** The Jacobian matrix of $\vec{f}(x, \vec{u}) = \begin{pmatrix} u_2 \\ g(x) - 3u_2 - 2u_1 \end{pmatrix}$ is

$$J = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}.$$

The entries are polynomials, hence everywhere continuous. Therefore, $\partial \vec{f}(x, \vec{u})/\partial \vec{u}$ is continuous in variables $x, \vec{u}$. 