Week 1 Examples

Three Examples. Solve differential equations without a book. Three basic examples used throughout a course in differential equations, which require only a calculus background.

Growth-Decay: \( \frac{dA(t)}{dt} = k A(t), \ A(0) = A_0. \)

The unique solution is \( A(t) = A_0 e^{kt}. \) Radioactive decay. Jeweler’s bench light experiment. Malthusian population dynamics. RC and LR circuits. Drug elimination. First-order chemical reactions, law of mass-action. Compound continuous bank interest.

Newton Cooling: \( \frac{du(t)}{dt} = -h(u(t) - u_1), \ u(0) = u_0. \)

The solution is \( u(t) = u_1 + (u_0 - u_1) e^{-ht}. \) Hot chocolate at initial temperature \( u_0 \) with room thermometer reading \( u_1. \) Symbol \( u(t) \) = time-varying hot chocolate dial thermometer temperature.

Verhulst Dynamics: \( \frac{dP(t)}{dt} = (a - b P(t)) P(t), \ P(0) = P_0. \)

The solution is \( P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}} \) Fish population \( P(t) \) in Crecet Lake at Alta. Carrying capacity. Stocking and re-stocking. Harvesting.

Example 1: Exercise 1.2-2: Solve \( dy/dx = (x - 2)^2, \ y(2) = 1. \)
Method of quadrature. Answer Check details. Non-reversible steps, false proof for \( 0 = 1. \)

Example 2: Exercise 1.2-4: \( dy/dx = 1/x^2, \ y(1) = 5. \)
Power rule in Newton calculus. Answer check shortcuts.

Example 3: Exercise 1.2-10: \( dy/dx = x e^{-x}, \ y(0) = 1. \)
Integral tables and integration by parts. Jennifer Lahti’s solution:

Example 4: Exercise 1.3-8: \( dy/dx = x^2 - y \)
Thread edge-to-edge solutions through the direction field at each blue dot. JPEG image source:

Example 5: Exercise 1.3-14: \( dy/dx = y^{1/3}, \ y(0) = 0 \)
Explain application of the Peano and Picard theorems.
Computer numerical methods fail on this example. Why?

Example 6: Exercise 1.4-6: Solve \( y' = 3\sqrt{xy} \)
Three answers. Book reports only one answer.

Example 7: Exercise 1.4-10: Solve \( (1 + x^2)y' = (1 + y)^2 \)
Two answers. Book reports only one answer.

Example 8: Exercise 1.4-18: Solve \( x^2y' = 1 - x^2 + y^2 - x^2y^2 \) [See Example 11 infra]

Example 9: Exercise 1.4-22: Solve \( y' = 4x^3y - y, \ y(1) = -3 \)

Example 10: Show that \( y' = x + y \) is not separable.
TEST I. \( f_x/f \) depends on \( y \) implies \( y' = f(x,y) \) not separable.

Example 11: Find a factorization \( f(x,y) = F(x)G(y) \) given
(1) \( f(x,y) = 2xy + 4y + 3x + 6 \)
(2) \( f(x,y) = (1 - x^2 + y^2 - x^2y^2)/x^2 \)
Answers: (1) \( F = x + 2, \ G = 2y + 3; \) (2) \( F = (1 - x^2)/x^2, \ G = 1 + y^2. \) Main idea: Choose \( y = 0 \) in \( F(x) = f(x,y)/G(y) \) to find \( F(x) = (3x + 6)/G(0) \) in equation (1). How to find \( G? \) Warning: Divide by zero is not allowed. Choose \( y = 0, \ y = 1, \) etc, until no divide by zero error.

Example 12: Midterm 1 examples: \( y' = x + y, \ y'_x = x + y^2, \ y'_y = x^2 + y^2 \)