

Numerical Methods for Differential Equations

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Rectangular Rule

The approximation uses Euler's idea of replacing the integrand by a constant. The value of the integral is approximately the area of a rectangle of width $b - a$ and height $F(a)$.

$$\int_a^b F(x) dx \approx (b - a)F(a)$$

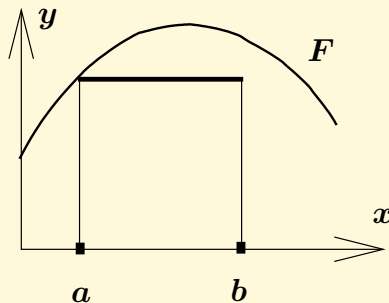


Figure 1. Rectangular Rule

Trapezoidal Rule

The rule replaces the integrand $F(x)$ by a linear function $L(x)$ which connects the planar points $(a, F(a))$, $(b, F(b))$. The value of the integral is approximately the area under the curve L , which is the area of a trapezoid.

$$\int_a^b F(x)dx \approx \frac{b-a}{2} (F(a) + F(b))$$

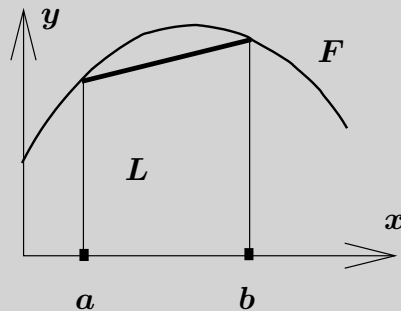


Figure 2. Trapezoidal Rule

Simpson's Rule

The rule replaces the integrand $F(x)$ by a quadratic polynomial $Q(x)$ which connects the planar points $(a, F(a))$, $((a+b)/2, F((a+b)/2))$, $(b, F(b))$. Then the integral of F is approximately the area under the quadratic curve Q .

$$\int_a^b F(x)dx \approx (b-a) \left(\frac{F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)}{6} \right)$$

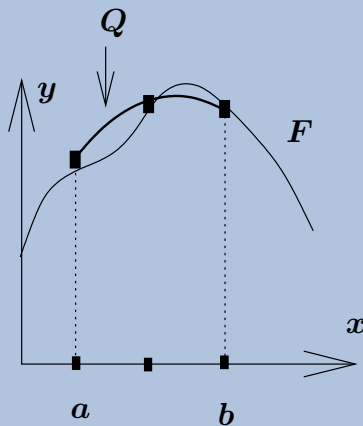
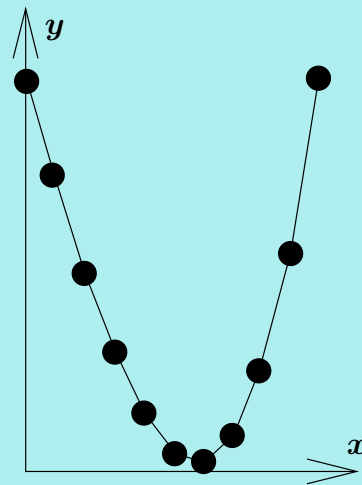


Figure 3. Simpson's Rule

How to make a connect-the-dots graphic

| x | y |
|-----|-------|
| 0.0 | 2.000 |
| 0.1 | 1.901 |
| 0.2 | 1.808 |
| 0.3 | 1.727 |
| 0.4 | 1.664 |
| 0.5 | 1.625 |

| x | y |
|-----|-------|
| 0.6 | 1.616 |
| 0.7 | 1.643 |
| 0.8 | 1.712 |
| 0.9 | 1.829 |
| 1.0 | 2.000 |



The table consists of xy -values for $y = x^3 - x + 2$. The graphic represents the table's rows, which are pairs (x, y) , as *dots*. Joined dots make the *connect-the-dots* graphic.

Maple code

A connect-the-dots graphic can be made in `maple` by supplying a list \mathbf{L} of pairs to be connected. An example:

```

$$\mathbf{L} := [[1, 3], [2, 1], [3, 5]]: \quad \text{plot}(\mathbf{L});$$

```

Numerical Methods for $y' = F(x)$

Quadrature applies to give $y(x) = y_0 + \int_{x_0}^x F(x)dx$. Numerical solution methods amount to approximating the integral on the right by Rectangular, Trapezoidal and Simpson methods.

The methods replace the exact value of an integral $\int_{x_0}^{x_0+h} F(x)dx$ by a numerical approximation value which is useful for graphics when h is small. Larger intervals are broken into smaller intervals of length h , then the approximation is applied.

Table 1. Three numerical integration methods.

| | |
|------|---------------------|
| Rect | $Y = y_0 + hF(x_0)$ |
|------|---------------------|

| | |
|------|--|
| Trap | $Y = y_0 + \frac{h}{2}(F(x_0) + F(x_0 + h))$ |
|------|--|

| | |
|------|--|
| Simp | $Y = y_0 + \frac{h}{6}(F(x_0) + 4F(x_0 + h/2) + F(x_0 + h))$ |
|------|--|

Maple code for the Rectangular and Trapezoid Rules

```
# Rectangular algorithm
# Group 1, initialize.
F:=x->evalf(cos(x) + 2*x):
x0:=0:y0:=0:h:=0.1*Pi:
Dots1:=[x0,y0]:
```

```
# Group 2, repeat 10 times
Y:=y0+h*F(x0):
x0:=x0+h:y0:=evalf(Y):
Dots1:=Dots1,[x0,y0];
```

```
# Group 3, plot.
plot([Dots1]);
```

```
# Trapezoidal algorithm
# Group 1, initialize.
F:=x->evalf(cos(x) + 2*x):
x0:=0:y0:=0:h:=0.1*Pi:
Dots2:=[x0,y0]:
```

```
# Group 2, repeat 10 times
Y:=y0+h*(F(x0)+F(x0+h))/2:
x0:=x0+h:y0:=evalf(Y):
Dots2:=Dots2,[x0,y0];
```

```
# Group 3, plot.
plot([Dots2]);
```

<http://www.math.utah.edu/~gustafso/s2016/2280/pdf/rect-algorithm.pdf>

<http://www.math.utah.edu/~gustafso/s2016/2280/pdf/trap-algorithm.pdf>

Maple code for Rectangular and Simpson Rules

```
# Rectangular algorithm
# Group 1, initialize.
F:=x->evalf(exp(-x*x)):
x0:=0:y0:=0:h:=0.1:
Dots1:=[x0,y0]:

# Group 2, repeat 10 times
Y:=evalf(y0+h*F(x0)):
x0:=x0+h:y0:=Y:
Dots1:=Dots1,[x0,y0]:

# Group 3, plot.
plot([Dots1]);
```

<http://www.math.utah.edu/~gustafso/s2016/2280/pdf/rect2-algorithm.pdf>

<http://www.math.utah.edu/~gustafso/s2016/2280/pdf/simp-algorithm.pdf>

```
# Simpson algorithm
# Group 1, initialize.
F:=x->evalf(exp(-x*x)):
x0:=0:y0:=0:h:=0.1:
Dots3:=[x0,y0]:

# Group 2, repeat 10 times
Y:=evalf(y0+h*(F(x0)+
    4*F(x0+h/2)+F(x0+h))/6):
x0:=x0+h:y0:=Y:
Dots3:=Dots3,[x0,y0]:

# Group 3, plot.
plot([Dots3]);
```


Numerical Methods for $y' = f(x, y)$

The methods replace the exact value of

$$y(x_0 + h) = y_0 + \int_{x_0}^{x_0+h} f(x, y(x)) dx$$

by a numerical approximation Y . The value is useful for graphics when h is small.

Table 2. Three numerical methods for $y' = f(x, y)$.

| | |
|-------|--|
| Euler | $Y = y_0 + hf(x_0, y_0)$ |
| Heun | $y_1 = y_0 + hf(x_0, y_0)$ |
| | $Y = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_0 + h, y_1))$ |
| RK4 | $k_1 = hf(x_0, y_0)$ |
| | $k_2 = hf(x_0 + h/2, y_0 + k_1/2)$ |
| | $k_3 = hf(x_0 + h/2, y_0 + k_2/2)$ |
| | $k_4 = hf(x_0 + h, y_0 + k_3)$ |
| | $Y = y_0 + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}$ |

Maple code for Euler and Heun methods

```
# Euler algorithm
# Group 1, initialize.
f:=(x,y)->-y+1-x:
x0:=0:y0:=3:h:=0.1:L:=[x0,y0]:

# Group 2, repeat 10 times
Y:=y0+h*f(x0,y0):
euler-algorithm.pdf
x0:=x0+h:y0:=Y:L:=L,[x0,y0]:

# Group 3, plot.
plot([L]);
```

<http://www.math.utah.edu/~gustafso/s2016/2280/pdf/euler-algorithm.pdf>

<http://www.math.utah.edu/~gustafso/s2016/2280/pdf/heun-algorithm.pdf>

```
# Heun algorithm
# Group 1, initialize.
f:=(x,y)->-y+1-x:
x0:=0:y0:=3:h:=0.1:L:=[x0,y0]:

# Group 2, repeat 10 times
Y:=y0+h*f(x0,y0):
Y:=y0+h*(f(x0,y0)+f(x0+h,Y))/2:
x0:=x0+h:y0:=Y:L:=L,[x0,y0]:

# Group 3, plot.
plot([L]);
```

Maple code for Heun and RK4 methods

```
# Heun algorithm
# Group 1, initialize.
f:=(x,y)->-y+1-x:
x0:=0:y0:=3:h:=0.1:L:=[x0,y0]:

# Group 2, repeat 10 times
Y:=y0+h*f(x0,y0):
Y:=y0+h*(f(x0,y0)+f(x0+h,Y))/2:
x0:=x0+h:y0:=Y:L:=L,[x0,y0];
```

```
# Group 3, plot.
plot([L]);
```

<http://www.math.utah.edu/~gustafso/s2016/2280/pdf/heun-algorithm.pdf>

<http://www.math.utah.edu/~gustafso/s2016/2280/pdf/rk4-algorithm.pdf>

```
# RK4 algorithm
# Group 1, initialize.
f:=(x,y)->-y+1-x:
x0:=0:y0:=3:h:=0.1:L:=[x0,y0]:

# Group 2, repeat 10 times.
k1:=h*f(x0,y0):
k2:=h*f(x0+h/2,y0+k1/2):
k3:=h*f(x0+h/2,y0+k2/2):
k4:=h*f(x0+h,y0+k3):
Y:=y0+(k1+2*k2+2*k3+k4)/6:
x0:=x0+h:y0:=Y:L:=L,[x0,y0];
```

```
# Group 3, plot.
plot([L]);
```

Numerical Algorithms: Planar Case

Notation. Let t_0, x_0, y_0 denote the entries of the dot table on a particular line. Let h be the increment for the dot table and let $t_0 + h, x, y$ stand for the dot table entries on the next line.

Planar Euler Method

$$\begin{aligned}x &= x_0 + hf(t_0, x_0, y_0), \\y &= y_0 + hg(t_0, x_0, y_0).\end{aligned}$$

Planar Heun Method

$$\begin{aligned}x_1 &= x_0 + hf(t_0, x_0, y_0), \\y_1 &= y_0 + hg(t_0, x_0, y_0), \\x &= x_0 + h(f(t_0, x_0, y_0) + f(t_0 + h, x_1, y_1))/2 \\y &= y_0 + h(g(t_0, x_0, y_0) + g(t_0 + h, x_1, y_1))/2.\end{aligned}$$

Planar RK4 Method

$$k_1 = hf(t_0, x_0, y_0),$$

$$m_1 = hg(t_0, x_0, y_0),$$

$$k_2 = hf(t_0 + h/2, x_0 + k_1/2, y_0 + m_1/2),$$

$$m_2 = hg(t_0 + h/2, x_0 + k_1/2, y_0 + m_1/2),$$

$$k_3 = hf(t_0 + h/2, x_0 + k_2/2, y_0 + m_2/2),$$

$$m_3 = hg(t_0 + h/2, x_0 + k_2/2, y_0 + m_2/2),$$

$$k_4 = hf(t_0 + h, x_0 + k_3, y_0 + m_3),$$

$$m_4 = hg(t_0 + h, x_0 + k_3, y_0 + m_3),$$

$$x = x_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4),$$

$$y = y_0 + \frac{1}{6} (m_1 + 2m_2 + 2m_3 + m_4).$$

Numerical Algorithms: General Case

Consider a vector initial value problem

$$\mathbf{u}'(t) = \mathbf{F}(t, \mathbf{u}(t)), \quad \mathbf{u}(t_0) = \mathbf{u}_0.$$

Vector Euler Method

$$\mathbf{u} = \mathbf{u}_0 + h\mathbf{F}(t_0, \mathbf{u}_0)$$

Vector Heun Method

$$\begin{aligned} \mathbf{w} &= \mathbf{u}_0 + h\mathbf{F}(t_0, \mathbf{u}_0), \\ \mathbf{u} &= \mathbf{u}_0 + \frac{h}{2} (\mathbf{F}(t_0, \mathbf{u}_0) + \mathbf{F}(t_0 + h, \mathbf{w})) \end{aligned}$$

Vector RK4 Method

$$\mathbf{k}_1 = h\mathbf{F}(t_0, \mathbf{u}_0),$$

$$\mathbf{k}_2 = h\mathbf{F}(t_0 + h/2, \mathbf{u}_0 + \mathbf{k}_1/2),$$

$$\mathbf{k}_3 = h\mathbf{F}(t_0 + h/2, \mathbf{u}_0 + \mathbf{k}_2/2),$$

$$\mathbf{k}_4 = h\mathbf{F}(t_0 + h, \mathbf{u}_0 + \mathbf{k}_3),$$

$$\mathbf{u} = \mathbf{u}_0 + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4).$$