Name: ________________

Differential Equations 2280
Final Exam
Thursday, 28 April 2017, 12:45pm-3:15pm

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

Chapters 1 and 2: Linear First Order Differential Equations

A **(a)** [60%] Solve $5v'(t) = 7 + \frac{4}{t+1}v(t)$, $v(0) = 7$. Show all integrating factor steps.

$v'(t) = \frac{7}{5} + \frac{4/5}{t+1}v(t)$, $v(0) = 7$

$v'(t) - \frac{4/5}{t+1}u(t) = \frac{7}{5}$ W = int. factor $= e^{\int \frac{-4/5}{t+1}} = e^{-4/5 \ln|t+1|} = |t+1|^{-4/5}$

$\int \frac{-4/5}{t+1} = -\frac{4}{5}\int\frac{1}{t+1} = -\frac{4}{5}\ln|t+1|$ $(t+1)^{-4/5}$

$(W v(t))' = \frac{7}{5}W$

quadrature →

$W v(t) = \frac{7}{5}\int W$ $\int W = \int (t+1)^{-4/5} = 5(t+1)^{1/5} + c$

→

$W v(t) = \frac{7}{5}\left(5(t+1)^{1/5} + c\right) = 7(t+1)^{1/5} + c$

$v(t) = \frac{7(t+1)^{1/5}}{(t+1)^{-4/5}} + \frac{c}{(t+1)^{-4/5}} = 7(t+1) + c(t+1)^{4/5}\Big|_{t=0}$

$= 7 \rightarrow c = 0$

$\boxed{v(t) = 7t + 7}$

(b) [20%] Solve the linear homogeneous equation $2\sqrt{x+2}\,\frac{dy}{dx} = 2xy$.

A

$$\frac{dy}{dx} - \frac{2x}{2\sqrt{x+2}}y = 0 \xrightarrow{\text{shortcut}} y = \frac{\text{const}}{\text{int fac}}.$$

$$\Downarrow$$

$$y' - \frac{x}{\sqrt{x+2}}y = 0 \qquad \text{int. fac.} = e^{-\int \frac{x}{\sqrt{x+2}}dx}$$

$$y = ce^{\int \frac{x}{\sqrt{x+2}}dx}$$

$$\int \frac{x}{\sqrt{x+2}}dx \qquad u = x+2, \quad du = dx \longrightarrow \int \frac{u-2}{\sqrt{u}}du$$

$$= \int \sqrt{u}\,du - 2\int u^{-1/2}du$$

$$= \frac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2}$$

$$\Downarrow$$

$$\boxed{y = ce^{\frac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2}}}$$

A **(c)** [20%] The linear problem $2\sqrt{x+2}\,y' = 2xy - 3x$ can be solved using superposition $y = y_h + y_p$. Find y_h and y_p.

y_p = equil. sol. $\rightarrow$ $2xy - 3x = x(2y-3) = 0 \rightarrow$

$$y_p = \frac{3}{2}$$

y_h from part b) $\rightarrow$

$$\boxed{y = ce^{\frac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2}} + \frac{3}{2}}$$

Chapter 3: Linear Equations of Higher Order

$(r^2-4r+4)+16=0$

$(r-2)^2+16=0$

(a) [10%] Solve for the general solution: $y'' - 4y' + 20y = 0$

A

char. eqn. $r^2-4r+20=0$

$r = 2 \pm \frac{1}{2}\sqrt{16-4(1)(20)}$

$= 2 \pm \frac{1}{2}\sqrt{-64} =$

$2 \pm 4i \rightarrow$

atoms $= e^{2t}\cos 4t,$

$e^{2t}\sin 4t$

$$\boxed{y = c_1 e^{2t}\cos 4t + c_2 e^{2t}\sin 4t}$$

(b) [20%] Solve for the general solution: $y^{(5)} + 289y^{(3)} = 0$

A

$\begin{array}{r} 17 \\ 17 \\ \hline 119 \\ 170 \\ \hline 289 \end{array}$

char. eqn. $r^5 + 289r^3 = 0 \rightarrow$

$r^3(r^2+289)=0$

$r = 0, 0, 0, \pm 17i$

atoms $= e^{0x}, xe^{0x}, x^2e^{0x}, \sin 17x, \cos 17x$

$$\boxed{y = c_1 + c_2 x + c_3 x^2 + c_4 \sin 17x + c_5 \cos 17x}$$

$r^2 - 4r + 20 = (r-2)^2 + 16$

(c) [20%] Solve for the general solution, given the characteristic equation is $r(r^3 - 4r)^2(r^2 - 4r + 20)^2 = 0$.

A

chr. eqn. $r^3 (r^2 - 4)^2 ((r-2)^2 + 16)^2 = 0$

$r = 0, 0, 0, \pm 2, \pm 2, 2 \pm 4i, 2 \pm 4i$

atoms $= 1, x, x^2, e^{2x}, e^{-2x}, xe^{2x}, xe^{-2x}, e^{2x}\cos 4x, e^{2x}\sin 4x, xe^{2x}\cos 4x, xe^{2x}\sin 4x$

$$\boxed{\begin{aligned} y &= c_1 + c_2 x + c_3 x^2 + c_4 e^{2x} + c_5 e^{-2x} + c_6 xe^{2x} + c_7 xe^{-2x} \\ &+ c_8 e^{2x}\cos 4x + c_9 e^{2x}\sin 4x + c_{10} xe^{2x}\cos 4x + c_{11} xe^{2x}\sin 4x \end{aligned}}$$

A **(d)** Given $\frac{1}{2}x''(t) + \frac{2}{5}x'(t) + \frac{2}{3}x(t) = 17\cos(\omega t)$, which represents a damped forced spring-mass system with $m = \frac{1}{2}$, $c = \frac{2}{5}$, $k = \frac{2}{3}$, answer the following two questions.

(1) [10%] Compute the frequency ω for practical mechanical resonance.

(2) [10%] Classify the homogeneous problem as over-damped, critically-damped or under-damped.

(1) $\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{\frac{4}{3} - \frac{(4/25)}{2(1/4)}} = \sqrt{\frac{4}{3} - \frac{(4/25)}{(1/2)}} = \sqrt{\frac{4}{3} - \frac{8}{25}}$

$= \sqrt{\frac{100}{75} - \frac{24}{75}} = \boxed{\sqrt{\frac{76}{75}}}$

(2) $\frac{1}{2}x'' + \frac{2}{5}x' + \frac{2}{3} = 0 \rightarrow r^2 + \frac{4}{5}r + \frac{4}{3} = 0$

$e^{-\cdots}\cos\ldots$

$\rightarrow$ oscillators

$\boxed{\text{underdamped}}$

$r = \frac{-4}{10} \pm \frac{1}{2}\sqrt{\frac{16}{25} - \frac{16}{3}}$

$e^{-\cdots}\sin\ldots$

negative, $\frac{48}{75} - \frac{400}{75} \approx \frac{-350}{75} \approx 5$

(e) [30%] Determine for $y^{(6)} - 4y^{(4)} = 5x^3 + x^2e^{2x} + \sin(2x)$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

A

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homogeneous problem: $y^{(6)} - 4y^{(4)} = 0$

chr. eqn. $r^6 - 4r^4 = 0$

$r^4(r^2-4) = 0$, $r = 0,0,0,0,\pm 2$

atoms $= 1, x, x^2, x^3, e^{2x}, e^{-2x}$

method of uc: first try $y_p = \underbrace{1 + x + x^2 + x^3}_{x^4} + \underbrace{e^{2x} + xe^{2x} + x^2e^{2x}}_{x} + \underbrace{\sin 2x + \cos 2}_{\text{nothing, okay}}$

no repeats - so must mult by:

$\rightarrow$ $y_p = d_1x^4 + d_2x^5 + d_3x^6 + d_4x^7 + d_5xe^{2x} + d_6x^2e^{2x} + d_7x^3e^{2x} + d_8\sin 2x + d_9\cos 2x$

Chapters 4 and 5: Systems of Differential Equations

(a) [20%] Matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}$ has eigenvalues $-1, 1, -5$. Find all eigenpairs of A and then write the solution of $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ according to the Eigenanalysis Method.

$\lambda_1 = -1$ $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix} \Rightarrow \overset{A}{\begin{pmatrix} +1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix}} \overset{\vec{v}_1}{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} = \vec{0}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\lambda_2 = 1$ $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -6 \end{pmatrix} \overset{\vec{v}_2}{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} = \vec{0}$

$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = -5$ $\begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \vec{0}$

$\begin{pmatrix} 5 & 1 & 1 \\ 0 & \frac{24}{5} & \frac{4}{5} \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{bmatrix} 5 & 1 & 1 \\ 0 & 24 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{4}{24} \\ 0 & 0 & 0 \end{pmatrix}$

$v_3 = \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}$

Check:

$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \lambda_1 \vec{v}_1$

$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \lambda_2 \vec{v}_2$

$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} -1+6 \\ -1+6 \\ -30 \end{pmatrix} = \lambda_3 \vec{v}_3$

$\begin{pmatrix} 5 \\ 5 \\ -30 \end{pmatrix}$

$-5 \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ -30 \end{pmatrix}$

$x_1 = -\frac{1}{6} x_3$ $\quad x_1 = -1$

$x_2 = -\frac{1}{6} x_3$ $\quad x_2 = -1$

$x_3 = x_3$ $\quad x_3 = 6 \quad x_3 = 6$

$$\boxed{\vec{x} = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t} \begin{pmatrix} -1 \\ -1 \\ 6 \end{pmatrix}}$$

(b) [30%] Find the general solution of the 2×2 system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

according to the Cayley-Hamilton-Ziebur Method, using the textbook's Chapter 4 shortcut.

$x'(t) = 5x(t) - y(t)$ A

$y'(t) = -x(t) + 5(y(t))$

$\underbrace{\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}}_{=A}$, $\det|A - \lambda I| = \begin{vmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{vmatrix}$

$= (5-\lambda)(5-\lambda) - 1 = 0$

$\lambda^2 - 10\lambda + 25 - 1 = 0$

$(\lambda - 5)^2 = 1$

$\lambda = 5 \pm 1$

$\lambda = 6$ or 4

~~atoms~~: $y = e^{rx}$

atoms: e^{6t}, e^{4t}

$x(t) = c_1 e^{6t} + c_2 e^{4t}$

$x'(t) - 5x(t) = -y(t)$

$y(t) = 5x - x'$

$x' = 6c_1 e^{6t} + 4c_2 e^{4t}$

$y(t) = 5c_1 e^{6t} + 5c_2 e^{4t} - 6c_1 e^{6t} - 4c_2 e^{4t}$

$-c_1 e^{6t} + c_2 e^{4t}$

$$\boxed{\begin{aligned} x(t) &= c_1 e^{6t} + c_2 e^{4t} \\ y(t) &= -c_1 e^{6t} + c_2 e^{4t} \end{aligned}}$$

(c) [20%] Assume a 2×2 system $\frac{d}{dt}\vec{u} = A\vec{u}$ has a scalar general solution

$$x(t) = c_1 e^{3t} + c_2 e^{4t}, \quad y(t) = 2c_2 e^{3t} + (c_1 + 3c_2)e^{4t}.$$

Compute the exponential matrix e^{At}.

A

$X(t) = c_1 e^{3t} + c_2 e^{4t}$

$Y(t) = c_1 e^{4t} + c_2(2e^{3t} + 3e^{4t})$

$$\Phi(t) = \begin{pmatrix} e^{3t} & e^{4t} \\ e^{4t} & 2e^{3t} + 3e^{4t} \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}$$

$$\Phi(0)^{-1} = \frac{1}{5-1}\begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix}$$

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{pmatrix} e^{3t} & e^{4t} \\ e^{4t} & 2e^{3t} + 3e^{4t} \end{pmatrix}\begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix}\left(\frac{1}{4}\right)$$

$$= \begin{pmatrix} 5e^{3t} - e^{4t} & -e^{3t} + e^{4t} \\ 5e^{4t} - (2e^{3t} + 3e^{4t}) & -e^{4t} + (2e^{3t} + 3e^{4t}) \end{pmatrix} \frac{1}{4}$$

$$e^{At} = \begin{pmatrix} \frac{5e^{3t} - e^{4t}}{4} & \frac{e^{4t} - e^{3t}}{4} \\ \frac{5}{4}e^{4t} - \frac{(2e^{3t} + 3e^{4t})}{4} & \frac{2e^{3t} + 3e^{4t} - e^{4t}}{4} \end{pmatrix}$$

(d) [30%] Consider the scalar system

$$\begin{cases} x' &= x \\ y' &= 3x + y, \\ z' &= x + z \end{cases}$$

Solve the system by the most efficient method.

$X' - X = 0$

$W_1 = I.F = e^{-t}$

$\boxed{y = C_1 e^t}$

$y' = 3c_1 e^t + y$

$y' - y = 3c_1 e^t$

$W_2 = e^{-t}$

$(y W_2)' = \int 3c_1$

$y W_2 = 3tc_1 + c_2$

$\boxed{y = 3te^t c_1 + c_2 e^t}$

$z' = c_1 e^t + z$

$z' - z = c_1 e^t$

$\frac{(W_3 z)'}{W} = c_1 e^t$

$(W_3 z)' = C_1$

$W_3 = e^{-t}$

$((W_3 z)' = \int c_1$

$W_3 z = C_1 t + C_3$

$\boxed{z = C_1 te^t + C_3 e^t}$

C

W

A

Chapter 6: Dynamical Systems

(a) [10%] Which of the four types *center, spiral, node, saddle* can be asymptotically stable at $t = \infty$? Explain your answer.

Spiral or node

saddle is never stable, center is always stable so no such thing as asymptotically stable w/ centers, and thus only be the ones spirals or nodes can ~~which can be asymptotically stable~~ spiral or node.

Thus by elimination only the spiral or node are canidates

(b) [20%] The origin is an equilibrium point of the linear system $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \mathbf{u}$. Classify $(0,0)$ as *center, spiral, node, saddle*.

$(1-\lambda)(2-\lambda)+1=0$

$2-\lambda-2\lambda+\lambda^2+1=0$

$\lambda^2-3\lambda+3=0$

$\lambda = \dfrac{3 \pm \sqrt{9-4(1)(3)}}{2}$

$= \dfrac{3}{2} \pm \dfrac{\sqrt{3}\, i}{2}$

$= \dfrac{3}{2} \pm \dfrac{\sqrt{3}i}{2}$

$\boxed{\text{Unstable spiral}}$

involves exponentials and sines, cosines

atoms: $e^{\frac{3}{2}t}\left(\cos\frac{\sqrt{3}}{2}t, \sin\frac{\sqrt{3}}{2}t\right)$

(c) [20%] Consider the nonlinear dynamical system

$$x' = 14x - \frac{1}{2}x^2 - xy, \quad y' = 16y - \frac{1}{2}y^2 - xy.$$

Find the equilibrium points.

Answer check: *One of the points is* $(0,32)$. *See part (d) below.*

$X(14 - \frac{1}{2}X - Y) = 0$

$Y(16 - \frac{1}{2}Y - X) = 0$

$(0, 32)$

$y(16 - \frac{1}{2}Y) = 0$

$(28, 0)$

$X(14 - \frac{1}{2}X) = 0$

$(0, 0)$

$14 - \frac{1}{2}X - Y = 0$

$16 - \frac{1}{2}Y - X = 0$

$\frac{1}{2}X + Y = 14$

$\frac{1}{2}Y + X = 16$

$X + 2Y = 28$

$2X + Y = 32$

$X = 28 - 2Y$

$2(28 - 2Y) + Y = 32$

$56 - 3Y = 32$

$-3Y = -24$

$Y = 8, X = 12$

$(12, 8)$

$\boxed{(0,32), (0,0), (28,0), (12,8)}$

equil.

(d) [30%] Consider again the nonlinear dynamical system

$$x' = 14x - \frac{1}{2}x^2 - xy, \quad y' = 16y - \frac{1}{2}y^2 - xy.$$

(1) Compute the linearization at equilibrium point $(0, 32)$ of this system, which is the **linear system** $\frac{d}{dt}\vec{u}(t) = J(0,32)\vec{u}(t)$.

(2) Classify the unique equilibrium $(0,0)$ of the linear system as a **node, spiral, center, saddle.**

(3) Report the equilibrium $(0,0)$ as **unstable** or **stable** at $t = \infty$. Classify further the equilibrium as a **repeller** or **attractor**, if the term applies.

A

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d. (1) $J(X,Y) = \begin{pmatrix} 14 - X - Y & -X \\ -Y & 16 - Y - X \end{pmatrix}$

$J(0,32) = \begin{pmatrix} -18 & 0 \\ -32 & -16 \end{pmatrix}$

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(2) $(-18 - r)(-16 - r) = 0$

$r = -16, -18$

Atoms: e^{-16t}, e^{-18t}

$\lim_{t \to \infty}$

$y(t) = e^{-16t} = 0$

$y(t) = e^{-18t} = 0$

$\boxed{\text{node}}$

(3) $\boxed{\text{Stable, attractor.}}$

(e) [20%] Consider again the nonlinear dynamical system

$$x' = 14x - \frac{1}{2}x^2 - xy, \quad y' = 16y - \frac{1}{2}y^2 - xy.$$

What classification can be deduced for equilibrium $(0, 32)$ of this nonlinear system, according to the Pasting Theorem (Theorem 2 in 6.2)? It is not enough to give a one-word classification. Please explain your answer fully and cite the applicability of the exceptions in the Pasting Theorem.

A

Since it is a node in the linear system with distinct roots it will also be a node in the non linear system

paste sys: node → node $r_1 \neq r_2$
spiral if $r_1 = r_2$

for nodes, when classifying a nonlinear system from the linear system classification using pasting theorem.

Chapter 7: Laplace Theory

(a) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s(s+2)^2}$.

$$\mathcal{L}\{f(t)\} = \frac{s+1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$= A + Be^{-2t} + Cte^{-2t}$$

$$s+1 = A(\cancel{s^2}+\cancel{4s}+4) + Bs(s+2) + C\cancel{s}$$
$$+ \cancel{B}(\cancel{s^2}+2\cancel{s})$$

$A + B = 0$ $4A + 2B + C = 1$ $4A = 1$

$1/4 = -B$ $1 - \frac{1}{2} + C = 1$ $A = 1/4$

$B = -1/4$ $-\frac{1}{2} + C = 0$

$C = 1/2$

$$f(t) = \frac{1}{4} - \frac{1}{4}e^{-2t} + \frac{1}{2}te^{-2t}$$

(b) [20%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^t \sin(2t)$.

$$f(t) = (-t)\, e^t \sin 2t)$$

$$\mathcal{L}\{f\} = \mathcal{L}\{-t\, e^t \sin 2t\}$$

$$= \frac{d}{ds}\mathcal{L}\{\sin 2t\}\Big|_{s \to s-1}$$

$$= \frac{d}{ds}\left(\frac{2}{s^2+4}\right)\Big|_{s \to s-1}$$

$$= \frac{d}{ds}\left(\frac{2}{(s-1)^2+4}\right)$$

$$= \frac{-2\,(2(s-1))}{((s-1)^2+4)^2} = \boxed{\frac{-4(s-1)}{((s-1)^2+4)^2}}$$

(c) [30%] Solve by Laplace's Method the forced linear dynamical system

$$\begin{cases} x' = x - y, \\ y' = x + y + 3, \end{cases}$$

subject to initial states $x(0) = 0$, $y(0) = 0$.

$x' = x - y \qquad x(0) = 0$

$sx(s) - \cancel{x(0)} = x(s) - y(s)$

$(s-1)x(s) + y(s) = 0$

$y' = x + y + 3 \qquad y(0) = 0$

$sy(s) - \cancel{y(0)} = x(s) + y(s) + \frac{3}{s}$

$-x(s) + (s-1)y(s) = 3/s$

use cramer's rule:

$D = \begin{bmatrix} s-1 & 1 \\ -1 & s-1 \end{bmatrix} \qquad x = \begin{bmatrix} 0 & 1 \\ 3/s & s-1 \end{bmatrix} \qquad y = \begin{bmatrix} s-1 & 0 \\ -1 & 3/s \end{bmatrix}$

$= (s-1)^2 + 1 \qquad = -\frac{3}{s} \qquad = \frac{3(s-1)}{s}$

$(s-1)(s-1)$

$s^2 - 2s + 1 + 1 \quad (s^2 - 2s + 2)$

$x(s) = \frac{-3}{s((s-1)^2+1)} \qquad y(s) = \frac{3(s-1)}{s((s-1)^2+1)}$

$\frac{-3}{s((s-1)^2+1)} = \frac{A}{s} + \frac{Bs}{s^2+1}\Big|_{s \to s-1} + \frac{C}{s^2+1}\Big|_{s \to s-1}$

y(s) partial fractions are same

$x(s) = A + Be^t\cos t + Ce^t\sin t \qquad y(s) = A + Be^t\cos t + Ce^t\sin t$

$-3 = A(\cancel{s^2} - \cancel{2s} + 2) + \cancel{B}s^2 + \cancel{C}s$

$A + B = 0 \qquad -2A + C = 0 \qquad 2A = -3$

$B = 3/2 \qquad +2(\cancel{+}\frac{3}{2}) + C = 0 \qquad A = -\frac{3}{2}$

$3 + C = 0$

$C = -3$

$3\cancel{s} - 3 = A(\cancel{s^2} - \cancel{2s} + 2) + B\cancel{s^2} + \cancel{C}s$

$A + B = 0 \qquad -2A + C = 3 \qquad 2A = -3$

$-\frac{3}{2} + B = 0 \qquad 3 + C = 3 \qquad A = -\frac{3}{2}$

$\qquad -3 \quad -3$

$B = 3/2 \qquad C = 0$

$x(s) = -\frac{3}{2} + \frac{3}{2}e^t\cos t - 3e^t\sin t$

$y(s) = -\frac{3}{2} + \frac{3}{2}e^t\cos t$

(d) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s}{s^2 - 4s + 20}$.

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2-4s+20} = \frac{s}{(s-2)^2+16}$$

$(s-\cancel{2})(s-2)$
s^2-4s+4

$$= \left.\frac{s}{u^2+16}\right|_{u=s-2} = \left.\frac{u+2}{u^2+16}\right|_{u=s-2} = \left.\frac{s+2}{s^2+16}\right|_{s=s-2}$$

$$= \left.\frac{s}{s^2+16}\right|_{s\to s-2} + \left.\frac{2}{s^2+16}\right|_{s\to s-2}$$

$$= e^{2t}\cos 4t + \frac{1}{2}e^{2t}\sin 4t$$

(e) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \frac{d}{ds}\left(\mathcal{L}\left(t^2 e^{4t}\cos t\right)\right)\Big|_{s\to s+1}.$$

$$\mathcal{L}\{f\} = \mathcal{L}\{-t\,t^2 e^{4t}\cos t\}\Big|_{s\to s+1}$$

$$= \mathcal{L}\{-t^3 e^{4t} e^{-t}\cos t\}$$

$$f = -t^3 e^{3t}\cos t$$

Chapter 9: Fourier Series and Partial Differential Equations

In parts (a) and (b), let $f_0(x) = 1$ on the interval $-2 < x < 0$, $f_0(x) = 0$ on the interval $0 < x < 2$, $f_0(x) = 0$ for $x = 0$ and $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(a) [10%] Compute the Fourier coefficients of $f(x)$ on $[-2, 2]$.

$$f_0(x) = \begin{cases} 1 & -2 < x < 0 \\ 0 & 0 < x < 2 \\ 0 & x = 0, \pm 2 \end{cases}$$

a) $a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{1}{2}\int_{-2}^{0} 1\,dt = \frac{1}{2}(t)\Big]_{-2}^{0} = \frac{1}{2}(0-(-2)) = \frac{1}{2}2 = 1$

for series $\frac{a_0}{2} + \sum \cdots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt = \frac{1}{2}\int_{-2}^{0}\cos\left(\frac{n\pi t}{2}\right)dt$$

$$= \frac{1}{2}\left(\frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right)\right)\Big]_{-2}^{0} = \frac{1}{2}(0-(0)) = \boxed{0}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt = \frac{1}{2}\int_{-2}^{0}\sin\left(\frac{n\pi t}{2}\right)dt$$

$$= \frac{1}{2}\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi t}{2}\right)\right]_{-2}^{0} = \frac{1}{2}\left[-\frac{2}{n\pi} - \left(-\frac{2}{n\pi}(-1)^n\right)\right]$$

$\cos(-\pi t)$

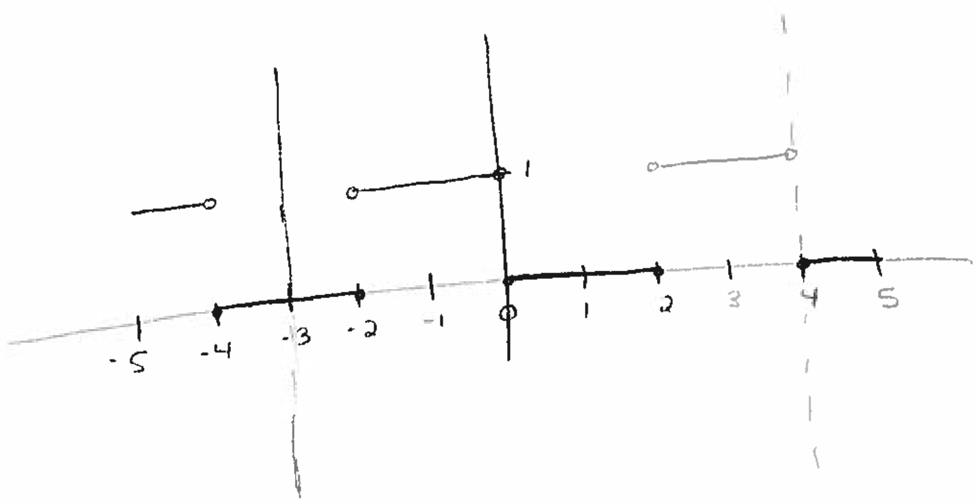
$$= \frac{1}{2}\left[-\frac{2}{n\pi} + \frac{2}{n\pi}(-1)^t\right] = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{n\pi}\right)\left[(-1)^n - 1\right]$$

$b_n = \frac{-2}{n\pi}$ For n odd, and $b_n = 0$ for n even

In parts (a) and (b), let $f_0(x) = 1$ on the interval $-2 < x < 0$, $f_0(x) = 0$ on the interval $0 < x < 2$, $f_0(x) = 0$ for $x = 0$ and $x = \pm 2$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 4.

(b) [10%] Graph $f(x)$ on $-5 < x < 5$. Find all values of x in $-3 \le x < 4$ which will exhibit Gibb's over-shoot.

$$f_0(x) = \begin{cases} 1 & -2 < x < 0 \\ 0 & 0 < x < 2 \\ 0 & x = 0, \pm 2 \end{cases}$$



Gibb's over-shoot wherever there is a jump discontinuity, so at $x = -2, 0, 2$ for the interval $[-3, 4)$

(c) [30%] **Heat Conduction in a Rod.** Solve the rod problem on $0 \le x \le 2$, $t \ge 0$:

$$\begin{cases} u_t &= 10u_{xx}, \\ u(0,t) &= 0, \\ u(2,t) &= 0, \\ u(x,0) &= 4\sin(6\pi x) + 15\sin(14\pi x) \end{cases}$$

$k = \sqrt{10}$ $\quad n=6$, $L=1$ $\quad$ add on $e^{-\left(\frac{n\pi}{L}\right)^2 kt}$ to each component

$$u(x,t) = 4\sin(6\pi x)\,e^{-(6\pi)^2\sqrt{10}\,t} + 15\sin(14\pi x)\,e^{-(14\pi)^2\sqrt{10}\,t}$$

(d) [40%] **Vibration of a Finite String.** The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ on $0 < x < L$, $t > 0$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes (the superposition of the normal modes). For a problem with shape $u(x,0) = 0$ and speed $u_t(x,0) = g(x)$, only the second normal modes $\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$ appear in the series solution.

Solve the finite string vibration problem on $0 \le x \le 5$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &= 100u_{xx}(x,t), & \\ u(0,t) &= 0, & \text{clamped at } x = 0 \\ u(4,t) &= 0, & \text{clamped at } x = 4 \\ u(x,0) &= 0, & \text{shape zero} \\ u_t(x,0) &= \sin(5\pi x) + 2\sin(7\pi x) & \text{speed nonzero} \end{cases}$$

For $u_t(x,0)$, use $\sin\left(\frac{n\pi ct}{L}\right)$. Here $c = 10$

$$u_t(x,0) = \sin(5\pi x) + 2\sin(7\pi x)$$

$$u_t(x,t) = \sin(5\pi x)\frac{\sin(5\pi\, 10t)}{50\pi} + \frac{2\sin(7\pi x)\sin(7\pi\, 10t)}{70\pi}$$

$$u_t(x,t) = \frac{\sin(5\pi x)\sin(50\pi t)}{50\pi} + \frac{2\sin(7\pi x)\sin(70\pi t)}{70\pi}$$

(over 50π and 70π because $u_t(x,t)$ is a derivative.)