Chapters 1 and 2: Linear First Order Differential Equations

(a) [60%] Solve $2v'(t) = 5 + \frac{1}{t+1}v(t)$, $v(0) = 5$. Show all integrating factor steps.

(b) [20%] Solve the linear homogeneous equation $2\sqrt{x+1} \frac{dy}{dx} = 2y$.

(c) [20%] The problem $2\sqrt{x+1} y' = 2y - 5$ can be solved using superposition $y = y_h + y_p$. Find $y_h$ and $y_p$. 
Chapter 3: Linear Equations of Higher Order

(a) [10%] Solve for the general solution: \( y'' + 4y' + 5y = 0 \)

(b) [20%] Solve for the general solution: \( y^{(6)} + 9y^{(4)} = 0 \)

(c) [20%] Solve for the general solution, given the characteristic equation is
\( r(r^3 + r)^2(r^2 + 2r + 17)^2 = 0 \).

(d) [20%] Given \( 6x''(t) + 2x'(t) + 2x(t) = 11 \cos(\omega t) \), which represents a damped forced spring-mass system with \( m = 6, c = 2, k = 2 \), answer the following questions.

True or False. Practical mechanical resonance occurs for input frequency \( \omega = \sqrt{11/6} \).

True or False. The homogeneous problem is over-damped.

(e) [30%] Determine for \( y^{(5)} + 4y^{(3)} = x^2 + e^x + \sin(2x) \) the shortest trial solution for \( y_p \) according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!
Chapters 4 and 5: Systems of Differential Equations

(a) [10%] Matrix \( A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix} \) has eigenpairs

\( \begin{pmatrix} -1, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} 1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} -5, \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix} \end{pmatrix} \).

Display the solution \( x(t) \) of \( x'(t) = Ax(t) \).

(b) [30%] Find the general solution of the 2 \( \times \) 2 system

\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}
\]

according to the Cayley-Hamilton-Ziebur Method, using the textbook’s shortcut in chapter 4.

(c) [10%] Assume a 2 \( \times \) 2 system \( \frac{d}{dt} \vec{u} = A\vec{u} \) has a scalar general solution

\[ x(t) = c_1 e^{-t} + c_2 e^{4t}, \quad y(t) = 4c_2 e^{-t} + (c_1 - 2c_2) e^{4t}. \]

Compute a fundamental matrix \( \Phi(t) \).

(d) [20%] Consider the scalar system

\[
\begin{cases} 
  x' &= x \\
  y' &= 3x, \\
  z' &= x + y
\end{cases}
\]

Solve the system by the most efficient method.
(a) [10%] The origin is an equilibrium point of the linear system \( u' = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} u \). Classify \((0, 0)\) as center, spiral, node, saddle.

In parts (b), (c), (d), consider the nonlinear dynamical system

\[
x' = 14x - 2x^2 - xy, \quad y' = 16y - 2y^2 - xy.
\]

(b) [20%] Find the equilibrium points for the nonlinear system (1).

(c) [30%] Consider again system (1). Classify the linearization at equilibrium point \((4, 6)\) as a node, spiral, center, saddle.

(d) [30%] Consider again system (1). What classification can be deduced for equilibrium \((4, 6)\) of this nonlinear system, according to the Pasting Theorem?
Chapter 7: Laplace Theory

(a) [10%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{1}{s(s + 1)^2}$.

(b) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t) \sinh(3t)$. This is the hyperbolic sine.

(c) [30%] Solve by Laplace’s Method the forced linear dynamical system

\[
\begin{align*}
x' &= x - y + 2, \\
y' &= x + y + 1,
\end{align*}
\]

subject to initial states $x(0) = 0, y(0) = 0$.

(d) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{s}{s^2 + 2s + 17}$.

(e) [10%] Solve for $f(t)$ in the relation

\[
\mathcal{L}(f) = \left(\mathcal{L}(t^2 e^{4t} \cos t)\right)\bigg|_{s \to s+2}.
\]
Chapter 9: Fourier Series and Partial Differential Equations

In parts (a) and (b), let \( f_0(x) = 1 \) on the interval \(-1 < x < 0\), \( f_0(x) = -1 \) on the interval \( 0 < x < 1\), \( f_0(x) = 0 \) for \( x = 0 \) and \( x = \pm 1\). Let \( f(x) \) be the periodic extension of \( f_0 \) to the whole real line, of period 2.

(a) \([10\%]\) Compute the Fourier coefficients of \( f(x) \) on \([-1, 1]\).

(b) \([10\%]\) Find all values of \( x \) in \(|x| < 3\) which will exhibit Gibb’s over-shoot.

(d) \([40\%]\) Heat Conduction in a Rod. Solve the rod problem on \( 0 \leq x \leq L, t \geq 0\):

\[
\begin{cases}
    u_t &= u_{xx}, \\
    u(0, t) &= 0, \\
    u(L, t) &= 0, \\
    u(x, 0) &= 5 \sin(2\pi x/L) + 12 \sin(4\pi x/L)
\end{cases}
\]

(e) \([30\%]\) Vibration of a Finite String. The normal modes for the string equation
\( u_{tt} = c^2 u_{xx} \) on \( 0 < x < L, t > 0 \) are given by the functions

\[
\sin \left( \frac{n\pi x}{L} \right) \cos \left( \frac{n\pi c t}{L} \right), \quad \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{n\pi c t}{L} \right).
\]

It is known that each normal mode is a solution of the string equation and that the problem below has solution \( u(x, t) \) equal to an infinite series of constants times normal modes (the superposition of the normal modes).

Solve the finite string vibration problem on \( 0 \leq x \leq 5, t > 0\):

\[
\begin{cases}
    u_{tt}(x, t) &= 25 u_{xx}(x, t), \\
    u(0, t) &= 0, \\
    u(5, t) &= 0, \\
    u(x, 0) &= \sin(5\pi x) + 2\sin(7\pi x), \\
    u_t(x, 0) &= 0
\end{cases}
\]