Differential Equations 2280
Final Exam
Thursday, 28 April 2016, 12:45pm-3:15pm

Instructions: This in-class exam is 120 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

Chapters 1 and 2: Linear First Order Differential Equations

(a) [60%] Solve $2v'(t) = 5 + \frac{1}{t+1}v(t)$, $v(0) = 5$. Show all integrating factor steps.

(b) [20%] Solve the linear homogeneous equation $2\sqrt{x+1}\,\frac{dy}{dx} = 2y$.

(c) [20%] The problem $2\sqrt{x+1}\,y' = 2y - 5$ can be solved using superposition $y = y_h + y_p$. Find y_h and y_p.

Chapter 3: Linear Equations of Higher Order

(a) [10%] Solve for the general solution: $y'' + 4y' + 5y = 0$

(b) [20%] Solve for the general solution: $y^{(6)} + 9y^{(4)} = 0$

(c) [20%] Solve for the general solution, given the characteristic equation is $r(r^3 + r)^2(r^2 + 2r + 17)^2 = 0$.

(d) [20%] Given $6x''(t) + 2x'(t) + 2x(t) = 11\cos(\omega t)$, which represents a damped forced spring-mass system with $m = 6$, $c = 2$, $k = 2$, answer the following questions.

True ☐ or False ☐. Practical mechanical resonance occurs for input frequency $\omega = \sqrt{11/6}$.

True ☐ or False ☐. The homogeneous problem is over-damped.

(e) [30%] Determine for $y^{(5)} + 4y^{(3)} = x^2 + e^x + \sin(2x)$ the shortest trial solution for y_p according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

Chapters 4 and 5: Systems of Differential Equations

(a) [10%] Matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -5 \end{pmatrix}$ has eigenpairs

$$\left(-1, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(-5, \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix}\right).$$

Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(b) [30%] Find the general solution of the 2×2 system

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

according to the Cayley-Hamilton-Ziebur Method, using the textbook's shortcut in chapter 4.

(c) [10%] Assume a 2×2 system $\frac{d}{dt}\vec{u} = A\vec{u}$ has a scalar general solution

$$x(t) = c_1 e^{-t} + c_2 e^{4t}, \quad y(t) = 4c_2 e^{-t} + (c_1 - 2c_2)e^{4t}.$$

Compute a fundamental matrix $\Phi(t)$.

(d) [20%] Consider the scalar system

$$\begin{cases} x' &= x \\ y' &= 3x, \\ z' &= x + y \end{cases}$$

Solve the system by the most efficient method.

Chapter 6: Dynamical Systems

(a) [10%] The origin is an equilibrium point of the linear system $\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u}$. Classify $(0,0)$ as *center, spiral, node, saddle.*

In parts (b), (c), (d), consider the nonlinear dynamical system

$$x' = 14x - 2x^2 - xy, \quad y' = 16y - 2y^2 - xy. \tag{1}$$

(b) [20%] Find the equilibrium points for the nonlinear system (1).

(c) [30%] Consider again system (1). Classify the linearization at equilibrium point $(4,6)$ as a node, spiral, center, saddle.

(d) [30%] Consider again system (1). What classification can be deduced for equilibrium $(4,6)$ of this nonlinear system, according to the Pasting Theorem?

Chapter 7: Laplace Theory

(a) [10%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s(s+1)^2}$.

(b) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t)\sinh(3t)$. This is the hyperbolic sine.

(c) [30%] Solve by Laplace's Method the forced linear dynamical system

$$\begin{cases} x' &= x - y + 2, \\ y' &= x + y + 1, \end{cases}$$

subject to initial states $x(0) = 0$, $y(0) = 0$.

(d) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s}{s^2 + 2s + 17}$.

(e) [10%] Solve for $f(t)$ in the relation

$$\mathcal{L}(f) = \left(\mathcal{L}\left(t^2 e^{4t}\cos t\right)\right)\Big|_{s \to s+2}.$$

Chapter 9: Fourier Series and Partial Differential Equations

In parts (a) and (b), let $f_0(x) = 1$ on the interval $-1 < x < 0$, $f_0(x) = -1$ on the interval $0 < x < 1$, $f_0(x) = 0$ for $x = 0$ and $x = \pm 1$. Let $f(x)$ be the periodic extension of f_0 to the whole real line, of period 2.

(a) [10%] Compute the Fourier coefficients of $f(x)$ on $[-1, 1]$.

(b) [10%] Find all values of x in $|x| < 3$ which will exhibit Gibb's over-shoot.

(d) [40%] **Heat Conduction in a Rod**. Solve the rod problem on $0 \le x \le L$, $t \ge 0$:

$$\begin{cases} u_t &= u_{xx}, \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= 5\sin(2\pi x/L) + 12\sin(4\pi x/L) \end{cases}$$

(e) [30%] **Vibration of a Finite String**. The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ on $0 < x < L$, $t > 0$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

It is known that each normal mode is a solution of the string equation and that the problem below has solution $u(x,t)$ equal to an infinite series of constants times normal modes (the superposition of the normal modes).

Solve the finite string vibration problem on $0 \le x \le 5$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &= 25u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(5,t) &= 0, \\ u(x,0) &= \sin(5\pi x) + 2\sin(7\pi x), \\ u_t(x,0) &= 0 \end{cases}$$