Differential Equations 2280

Sample Midterm Exam 3 with Solutions
Exam Date: Friday 14 April 2017 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exams 1, 2.

Chapter 3

1. **(Linear Constant Equations of Order n)**

(a) Find by variation of parameters a particular solution y_p for the equation $y'' = 1 - x$. Show all steps in variation of parameters. Check the answer by quadrature.

(b) A particular solution of the equation $mx'' + cx' + kx = F_0 \cos(2t)$ happens to be $x(t) = 11\cos 2t + e^{-t}\sin\sqrt{11}t - \sqrt{11}\sin 2t$. Assume m, c, k all positive. Find the unique periodic steady-state solution x_{SS}.

(c) A fourth order linear homogeneous differential equation with constant coefficients has two particular solutions $2e^{3x} + 4x$ and xe^{3x}. Write a formula for the general solution.

(d) Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 64x = 40\cos(4t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time, don't convert to phase-amplitude form**.

(e) Write the solution $x(t)$ of

$$x''(t) + 25x(t) = 180\sin(4t), \quad x(0) = x'(0) = 0,$$

as the sum of two harmonic oscillations of different natural frequencies.
To save time, don't convert to phase-amplitude form.

(f) Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 2x = 5\sin(t)$.

(g) Given $5x''(t) + 2x'(t) + 4x(t) = 0$, which represents a damped spring-mass system with $m = 5$, $c = 2$, $k = 4$, determine if the equation is over-damped , critically damped or under-damped.
To save time, do not solve for $x(t)$!

(h) Determine the practical resonance frequency ω for the electric current equation

$$2I'' + 7I' + 50I = 100\omega\cos(\omega t).$$

(i) Given the forced spring-mass system $x'' + 2x' + 17x = 82\sin(5t)$, find the steady-state periodic solution.

(j) Let $f(x) = x^3 e^{1.2x} + x^2 e^{-x}\sin(x)$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation of least order which has $f(x)$ as a solution. To save time, do not expand the polynomial and do not find the differential equation.

Use this page to start your solution.

Chapters 4 and 5

2. **(Systems of Differential Equations)**
Background. Let A be a real 3×3 matrix with eigenpairs $(\lambda_1, \mathbf{v}_1)$, $(\lambda_2, \mathbf{v}_2)$, $(\lambda_3, \mathbf{v}_3)$. The eigenanalysis method says that the 3×3 system $\mathbf{x}' = A\mathbf{x}$ has general solution

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}.$$

Background. Let A be an $n \times n$ real matrix. The method called **Cayley-Hamilton-Ziebur** is based upon the result

> The components of solution $\mathbf{x}$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$.

Background. Let A be an $n\times n$ real matrix. An augmented matrix $\Phi(t)$ of n independent solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is called a **fundamental matrix**. It is known that the general solution is $\mathbf{x}(t) = \Phi(t)\mathbf{c}$, where $\mathbf{c}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, $|\Phi(0)| \ne 0$.

(a) Display eigenanalysis details for the 3×3 matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix},$$

then display the general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(b) The 3×3 triangular matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix},$$

represents a linear cascade, such as found in brine tank models. Using the linear integrating factor method, starting with component $x_3(t)$, find the vector general solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(c) The exponential matrix e^{At} is defined to be a fundamental matrix $\Psi(t)$ selected such that $\Psi(0) = I$, the $n\times n$ identity matrix. Justify the formula $e^{At} = \Phi(t)\Phi(0)^{-1}$, valid for *any* fundamental matrix $\Phi(t)$.

(d) Let A denote a 2×2 matrix. Assume $\mathbf{x}'(t) = A\mathbf{x}(t)$ has scalar general solution $x_1 = c_1 e^t + c_2 e^{2t}$, $x_2 = (c_1 - c_2)e^t + 2c_1 + c_2)e^{2t}$, where c_1, c_2 are arbitrary constants. Find a fundamental matrix $\Phi(t)$ and then go on to find e^{At} from the formula in part (c) above.

(e) Let A denote a 2×2 matrix and consider the system $\mathbf{x}'(t) = A\mathbf{x}(t)$. Assume fundamental matrix $\Phi(t) = \begin{pmatrix} e^t & e^{2t} \\ 2e^t & -e^{2t} \end{pmatrix}$. Find the 2×2 matrix A.

(f) The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = 3x + y, \quad y' = -x + 3y,$$

which has complex eigenvalues $\lambda = 3 \pm i$. Show the details of the method, then go on to report a fundamental matrix $\Phi(t)$.
Remark. The vector general solution is $\mathbf{x}(t) = \Phi(t)\mathbf{c}$, which contains no complex numbers. Reference: 4.1, Examples 6,7,8.

Use this page to start your solution.

Chapter 6

3. (Linear and Nonlinear Dynamical Systems)

(a) Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} 3 & 4 \\ -2 & -1 \end{pmatrix} \vec{u}$$

(b) Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node.

$$\vec{u}' = \begin{pmatrix} -3 & 2 \\ -4 & 1 \end{pmatrix} \vec{u}$$

(c) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - y + 32, \\ y' &= 2x^2 - 2xy. \end{aligned}$$

An equilibrium point is $x = 4$, $y = 4$. Compute the Jacobian matrix $A = J(4, 4)$ of the linearized system at this equilibrium point.

(d) Consider the nonlinear dynamical system

$$\begin{aligned} x' &= -x - 2y^2 - y + 32, \\ y' &= 2x^2 + 2xy. \end{aligned}$$

An equilibrium point is $x = -4$, $y = 4$. Compute the Jacobian matrix $A = J(-4, 4)$ of the linearized system at this equilibrium point.

(e) Consider the nonlinear dynamical system $\left\{ \begin{aligned} x' &= -4x + 4y + 9 - x^2, \\ y' &= 3x - 3y. \end{aligned} \right.$

At equilibrium point $x = 3$, $y = 3$, the Jacobian matrix is $A = J(3, 3) = \begin{pmatrix} -10 & 4 \\ 3 & -3 \end{pmatrix}$.

> **(1)** Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the linear system $\frac{d}{dt}\vec{u} = A\vec{u}$.
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> **(2)** Apply the Pasting Theorem to classify $x = 3$, $y = 3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

(f) Consider the nonlinear dynamical system $\left\{ \begin{aligned} x' &= -4x - 4y + 9 - x^2, \\ y' &= 3x + 3y. \end{aligned} \right.$

At equilibrium point $x = 3$, $y = -3$, the Jacobian matrix is $A = J(3, -3) = \begin{pmatrix} -10 & -4 \\ 3 & 3 \end{pmatrix}$.

> **Linearization**. Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\dfrac{d}{dt}\vec{u} = A\vec{u}$.
>
> **Nonlinear System**. Apply the Pasting Theorem to classify $x = 3$, $y = -3$ as a saddle, center, spiral or node for the **nonlinear dynamical system**. Discuss all details of the application of the theorem. *Details count 75%.*

Use this page to start your solution.

Final Exam Problems

Chapter 5. Solve a homogeneous system $u' = Au$, $u(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ using the matrix exponential, Zeibur's method, Laplace resolvent and eigenanalysis.

Chapter 5. Solve a non-homogeneous system $u' = Au + F(t)$, $u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $F(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ using variation of parameters.