Differential Equations 2280

Midterm Exam 3

Exam Date: 22 April 2016 at 12:50pm

100%

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Chapter 3

100

1. **(Linear Constant Equations of Order n)**

A (a) [30%] Find by variation of parameters a particular solution y_p for the equation $y'' = x^2$. Show all steps in variation of parameters. Check the answer by quadrature.

A (b) [40%] Find the **Beats** solution for the forced undamped spring-mass problem

$$x'' + 256x = 231\cos(5t), \quad x(0) = x'(0) = 0.$$

It is known that this solution is the sum of two harmonic oscillations of different frequencies. **To save time,** please don't convert to phase-amplitude form.

A (c) [20%] Given $mx''(t) + cx'(t) + kx(t) = 0$, which represents a damped spring-mass system, assume $m = 9$, $c = 24$, $k = 16$. Determine if the equation is over-damped , critically damped or under-damped. **To save time,** do not solve for $x(t)$.

A ~~skip to~~ (d) [10%] Determine the practical resonance frequency ω for the RLC current equation

$$2I'' + 7I' + 50I = 500\sin(\omega t).$$

(1a) $y'' = x^2 \qquad f(x) = x^2$

$y'' = 0 \to r^2 = 0 \to y_h = c_1 + c_2 x$, $\quad y_1 = 1$, $y_2 = x$

$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$

$y = -y_1 \int \frac{y_2 f(x)}{W}\,dx + y_2 \int \frac{y_1 f(x)}{W}\,dx$

$y = -1 \int \frac{x \cdot x^2}{1}\,dx + x \int \frac{1 \cdot x^2}{1}\,dx = -\int x^3\,dx + x\int x^2\,dx = -\frac{x^4}{4} + x \cdot \frac{x^3}{3}$

$= -\frac{x^4}{4} + \frac{x^4}{3}$

$= -\frac{3x^4}{12} + \frac{4x^4}{12}$

$= \boxed{\frac{x^4}{12}}$

check: $\iint x^2 = \int \frac{x^3}{3} = \frac{x^4}{12}$ ✓

Use this page to start your solution.

b) $x'' + 256x = 231\cos 5t \qquad x(0) = x'(0) = 0$

Trial solution: $x = d_1 \cos 5t + d_2 \sin 5t$

$x' = -5d_1 \sin 5t + 5d_2 \cos 5t$

$x'' = -25d_1 \cos 5t - 25d_2 \sin 5t$

$-25d_1 \cos 5t - 25d_2 \sin 5t + 256d_1 \cos 5t + 256 d_2 \sin 5t = 231 \cos 5t$

$231 d_1 \cos 5t + 231 d_2 \sin 5t = 231 \cos 5t$

$\Rightarrow d_1 = 1,\ d_2 = 0$

$x = \cos 5t$

homogeneous:

$x'' + 256x = 0 \rightarrow r^2 + 256 = 0$

$r = \pm 16i$

$x_h = C_1 \cos 16t + C_2 \sin 16t$

$x = C_1 \cos 16t + C_2 \sin 16t + \cos 5t \rightarrow 0 = C_1 \cdot 1 + 1 \rightarrow C_1 = -1$

$x' = -16C_1 \sin 16t + 16C_2 \cos 16t - 5\sin 5t \rightarrow 0 = 16 \cdot C_2 \rightarrow C_2 = 0$

$\boxed{x = -\cos 16t + \cos 5t}$

1c) $9x'' + 24x' + 16x = 0 \qquad \sqrt{24^2 - 4(9)(16)} = \sqrt{576 - 576} = 0$

$\boxed{\text{critically damped}}$

1d) $2I'' + 7I' + 50I = 500\sin(wt)$

$w = \dfrac{1}{\sqrt{LC}}, \qquad L = 2,\ R = 7 \qquad \frac{1}{C} = 50,\ \text{so } C = \frac{1}{50}$

$= \dfrac{1}{\sqrt{2/50}} = \sqrt{\dfrac{50}{2}} = \sqrt{25} = \boxed{5 = w}$

Chapters 4 and 5

2. (Systems of Differential Equations)

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A **(a)** [30%] The 3×3 matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

has eigenvalues $\lambda = 3, 4, 5$. One Euler solution vector is $\vec{v}e^{\lambda t}$ with $\lambda = 3$ and $\vec{v} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Find two more Euler solution vectors and then display the vector general solution $\vec{x}(t)$ of $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

A **(b)** [40%] The 3×3 triangular matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix},$$

represents a linear cascade, such as found in brine tank models.

Part 1. Use the linear integrating factor method to find the vector general solution $\vec{x}(t)$ of $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$.

Part 2. The eigenanalysis method fails for this example. Cite two different methods, besides the linear integrating factor method, which apply to solve the system $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$. Don't show solution details for these methods, but explain precisely each method and why the method applies.

A **(c)** [30%] The Cayley-Hamilton-Ziebur shortcut applies especially to the system

$$x' = x + 4y, \quad y' = -4x + y,$$

which has complex eigenvalues $\lambda = 1 \pm 4i$.

Part 1. Show the details of the method, finally displaying formulas for $x(t), y(t)$.

Part 2. Report a fundamental matrix $\Phi(t)$.

(2a) $\lambda = 4$: $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad a+b=0,\ a+c=0,\ 0=0$ $\quad$ choose $c=1$ then $a=-1$ & $b=1$ $\quad \lambda=4: \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 5$: $A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad -a+b+c=0,\ a-b+c=0,\ -c=0,\ c=0$ $\quad$ then $a=b=1$ $\quad \lambda=5: \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Use this page to start your solution.

$$\boxed{\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{5t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

(2) Part 1

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} \qquad \begin{aligned} x' &= 3x + y \\ y' &= 3y + z \\ z' &= 4z \end{aligned}$$

$$z' - 4z = 0 \Rightarrow z = \frac{c_1}{e^{-4t}} = c_1 e^{4t}$$

$$y' - 3y = z \Rightarrow y' - 3y = c_1 e^{4t} \qquad W = \text{int factor} = e^{-3t}$$

$$\frac{(Wy)'}{W} = c_1 e^{4t}$$

$$(Wy)' = c_1 W e^{4t}$$

$$\int (Wy)' = \int c_1 e^{-3t} e^{4t}\, dt = \int c_1 e^t\, dt = c_1 e^t + c_2$$

$$\Downarrow$$

$$y = \frac{c_1 e^t}{e^{-3t}} + \frac{c_2}{e^{-3t}} = c_1 e^{4t} + c_2 e^{3t}$$

$$x' - 3x = y = c_1 e^{4t} + c_2 e^{3t} \qquad W = e^{-3t}$$

$$\int (Wx)' = \int (c_1 W e^{4t} + c_2 W e^{3t})\, dt = \int (c_1 e^{-3t} e^{4t} + c_2 e^{-3t} e^{3t})\, dt$$

$$= \int (c_1 e^t + c_2)\, dt$$

$$Wx = c_1 e^t + c_2 t + c_3$$

$$x = c_1 \underbrace{e^t e^{3t}}_{= e^{4t}} + c_2 t e^{3t} + c_3 e^{3t}$$

$$\boxed{\begin{aligned} x &= c_1 e^{4t} + c_2 t e^{3t} + c_3 e^{3t} \\ y &= c_1 e^{4t} + c_2 e^{3t} \\ z &= c_1 e^{4t} \end{aligned}}$$

Part 2

1) The Cayley-Hamilton Ziebur method can apply to a 3x3 system, details of the method are shown for part c of this problem.

2) Laplace transforms could be used by starting with z' equation and working upwards.

2c $x' = x + 4y, \quad y' = -4x + y$ $\quad \lambda = 1 \pm 4i$

$$A = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix} \to \begin{pmatrix} 1-\lambda & 4 \\ -4 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 16 = 0$$

$$(1-\lambda)^2 = -16$$

$$1 - \lambda = \pm 4i$$

$$\lambda = 1 \pm 4i$$

$$\boxed{x = c_1 e^t \cos 4t + c_2 e^t \sin 4t}$$

$$x' = x + 4y \to y = \frac{1}{4}(x' - x)$$

$$x' = \underbrace{c_1 e^t \cos 4t \; c_2 e^t \sin 4t}_{x} - 4c_1 e^t \sin 4t + 4c_2 e^t \cos 4t$$

so $y = \frac{1}{4}\left(x - 4c_1 e^t \sin 4t + 4c_2 e^t \cos 4t - x\right)$

$$\boxed{y = -c_1 e^t \sin 4t + c_2 e^t \cos 4t}$$

$$\Phi(t) = \begin{pmatrix} e^t \cos 4t & e^t \sin 4t \\ -e^t \sin 4t & e^t \cos 4t \end{pmatrix}$$

Chapter 6

100

3. (Linear and Nonlinear Dynamical Systems)

A **(a)** [20%] Determine whether the unique equilibrium $\vec{u} = \vec{0}$ is stable or unstable. Then classify the equilibrium point $\vec{u} = \vec{0}$ as a saddle, center, spiral or node. Sub-classification into improper or proper node is not required.

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} \vec{u}$$

A **(b)** [30%] Consider the nonlinear dynamical system

$$\begin{aligned} x' &= x - 2y^2 - 2y + 32, \\ y' &= 2x(x - 2y). = 2x^2 - 4xy \end{aligned}$$

An equilibrium point is $x = -8$, $y = -4$. Compute the Jacobian matrix of the linearized system at this equilibrium point.

A **(c)** [30%] Consider the soft nonlinear spring system $\left\{ \begin{aligned} x' &= y, \\ y' &= -5x - 2y + \frac{5}{4}x^3. \end{aligned} \right.$

(1) Determine the stability at $t = \infty$ and the phase portrait classification saddle, center, spiral or node at $\vec{u} = \vec{0}$ for the **linear dynamical system** $\frac{d}{dt}\vec{u} = A\vec{u}$, where A is the Jacobian matrix of this system at $x = 2$, $y = 0$.

(2) Apply the Pasting Theorem to classify $x = 2$, $y = 0$ as a saddle, center, spiral or node for the **nonlinear dynamical system.** Discuss all details of the application of the theorem. *Details count 75%.*

(d) [20%] State the hypotheses and the conclusions of the *Pasting Theorem* used in part (c) above. Accuracy and completeness expected.

3a $\begin{pmatrix} -1-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} = (-1-\lambda)(1-\lambda)+2 = -1-\lambda+\lambda+\lambda^2+2 = \lambda^2+1 \rightarrow \lambda = \pm i$

$\lambda = \pm i$, therefore $\vec{u} = \vec{0}$ is $\boxed{\text{stable and is a center}}$

3b $J(x,y) = \begin{pmatrix} 1 & -4y-2 \\ 4x-4y & -4x \end{pmatrix}$, $J(-8,-4) = \begin{pmatrix} 1 & 16-2 \\ -16+32 & 32 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 14 \\ 16 & 32 \end{pmatrix}}$

Use this page to start your solution.

3c) $\begin{cases} x' = y \\ y' = -5x - 2y + \frac{5}{4}x^3 \end{cases}$

① $J(x,y) = \begin{pmatrix} 0 & 1 \\ -5+\frac{15}{4}x^2 & -2 \end{pmatrix}$ $\qquad J(2,0) = \begin{pmatrix} 0 & 1 \\ 10 & -2 \end{pmatrix}$

$-5 + \frac{15}{4} \cdot 4 = 10$

$\begin{pmatrix} 0-\lambda & 1 \\ 10 & -2-\lambda \end{pmatrix} = -\lambda(-2-\lambda) - 10 = 2\lambda + \lambda^2 - 10 = 0$

$\lambda^2 + 2\lambda - 10 = 0$

$(\lambda+1)^2 - 11 = 0$

$\lambda = -1 \pm \sqrt{11}$

↳ eigenvalues real, opposite signs

⇒ saddle, unstable

Pasting thm implies $(2,0)$ as saddle for nonlinear dynamical system

3d) Pasting theorem says for classifications of critical points for non-Linear system:

① If $\lambda_1 = \lambda_2$ and λ_1, λ_2 real eigenvalues then $\begin{cases} \lambda_1, \lambda_2 < 0 \text{ stable} \\ \lambda_1, \lambda_2 > 0 \text{ unstable} \end{cases}$ and critical pt will be a node or a spiral

② If $\lambda_1, \lambda_2 = \pm bi$, pure imaginary eigenvalues, then critical point will be a center or a spiral and will be stable or unstable

③ If λ_1, λ_2 are not as above, then the linear classifications will be true for the non-linear system