Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exam 1.

1. (Chapter 3)
   (a) [50%] Find by any applicable method the steady-state periodic solution for the current equation
   \[ I'' + 2I' + 5I = -10 \sin (t). \]
   (b) [50%] Find by variation of parameters a particular solution \( y_p \) for the equation \( y'' = 1 - x. \) Show all steps in variation of parameters. Check the answer by quadrature.

2. (Chapters 1, 2, 3)
   (2a) [20%] Solve \( 2v'(t) = -8 + 2t + 1 v(t), \ v(0) = -4. \) Show all integrating factor steps.
   (2b) [10%] Solve for the general solution: \( y'' + 4y' + 6y = 0. \)
   (2c) [10%] Solve for the general solution of the homogeneous constant-coefficient differential equation whose characteristic equation is \( r(r^2 + r)^2(r^2 + 9)^2 = 0. \)
   (2d) [20%] Find a linear homogeneous constant coefficient differential equation of lowest order which has a particular solution \( y = x + \sin \sqrt{2}x + e^{-x} \cos 3x. \)
   (2e) [15%] A particular solution of the equation \( mx'' + cx' + kx = F_0 \cos(2t) \) happens to be \( x(t) = 11 \cos 2t + e^{-t} \sin \sqrt{11}t - \sqrt{11} \sin 2t. \) Assume \( m, c, k \) all positive. Find the unique periodic steady-state solution \( x_{ss}. \)
   (2f) [25%] Determine for \( y''' + y'' = 100x^2 + \sin x \) the shortest trial solution for \( y_p \) according to the method of undetermined coefficients. Do not evaluate the undetermined coefficients!

3. (Laplace Theory)
   (a) [50%] Solve by Laplace’s method \( x'' + 2x' + x = e^t, \ x(0) = x'(0) = 0. \)
   (b) [25%] Assume \( f(t) \) is of exponential order. Find \( f(t) \) in the relation
   \[ \left. \frac{d}{ds} \mathcal{L}(f(t)) \right|_{s \to (s-3)} = \mathcal{L}(f(t) - t). \]
   (c) [25%] Derive an integral formula for \( y(t) \) by Laplace transform methods, explicitly using the Convolution Theorem, for the problem
   \[ y''(t) + 4y'(t) + 4y(t) = f(t), \quad y(0) = y'(0) = 0. \]
   This is similar to a required homework problem from Chapter 7.

4. (Laplace Theory)
   (4a) [20%] Explain Laplace’s Method, as applied to the differential equation \( x'(t) + 2x(t) = e^t, \ x(0) = 1. \) Reference only. Not to appear on any exam.
   (4b) [15%] Solve \( \mathcal{L}(f(t)) = \frac{100}{(s^2 + 1)(s^2 + 4)} \) for \( f(t). \)
   (4c) [15%] Solve for \( f(t) \) in the equation \( \mathcal{L}(f(t)) = \frac{1}{s^2(s + 3)}. \)
   (4d) [10%] Find \( \mathcal{L}(f) \) given \( f(t) = (-t)e^{2t} \sin(3t). \)
   (4e) [20%] Solve \( x'' + x'' = 0, \ x(0) = 1, \ x'(0) = 0, \ x''(0) = 0 \) by Laplace’s Method.
   (4f) [20%] Solve the system \( x' = x + y, \ y' = x - y + 2, \ x(0) = 0, \ y(0) = 0 \) by Laplace’s Method.
5. (Laplace Theory)
   (a) [30%] Solve \( \mathcal{L}(f(t)) = \frac{1}{(s^2 + s)(s^2 - s)} \) for \( f(t) \).
   (b) [20%] Solve for \( f(t) \) in the equation \( \mathcal{L}(f(t)) = \frac{s + 1}{s^2 + 4s + 5} \).
   (c) [20%] Let \( u(t) \) denote the unit step. Solve for \( f(t) \) in the relation
   \[ \mathcal{L}(f(t)) = \frac{d}{ds} \mathcal{L}(u(t - 1) \sin 2t) \]
   Remark: This is not a second shifting theorem problem.
   (d) [30%] Compute \( \mathcal{L}(e^{2t} f(t)) \) for \( f(t) = e^{t} - e^{-t} \).

6. (Systems of Differential Equations)
The eigenanalysis method says that, for a \( 3 \times 3 \) system \( \mathbf{x}' = A \mathbf{x} \), the general solution is
   \( \mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + c_3 \mathbf{v}_3 e^{\lambda_3 t} \). In the solution formula, \((\lambda_i, \mathbf{v}_i), i = 1, 2, 3, \) is an eigenpair of \( A \). Given
   \[ A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \]
   then
   (a) [75%] Display eigenanalysis details for \( A \).
   (b) [25%] Display the solution \( \mathbf{x}(t) \) of \( \mathbf{x}'(t) = A \mathbf{x}(t) \).
   (c) Repeat (a), (b) for the matrix \( A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix} \).

7. (Systems of Differential Equations)
   (a) [30%] Find the eigenvalues of the matrix \( A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix} \).
   (b) [20%] Justify that eigenvectors of \( A \) corresponding to the eigenvalues in increasing order are the four vectors
   \[ \begin{bmatrix} 1 \\ -5 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \]
   (c) [50%] Display the general solution of \( \mathbf{u}' = A \mathbf{u} \) according to the Eigenanalysis method.

8. (Systems of Differential Equations)
   (a) [100%] The eigenvalues are 3, 5 for the matrix \( A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \).
   Display the general solution of \( \mathbf{u}' = A \mathbf{u} \) according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5). Assume initial condition \( \mathbf{u}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).