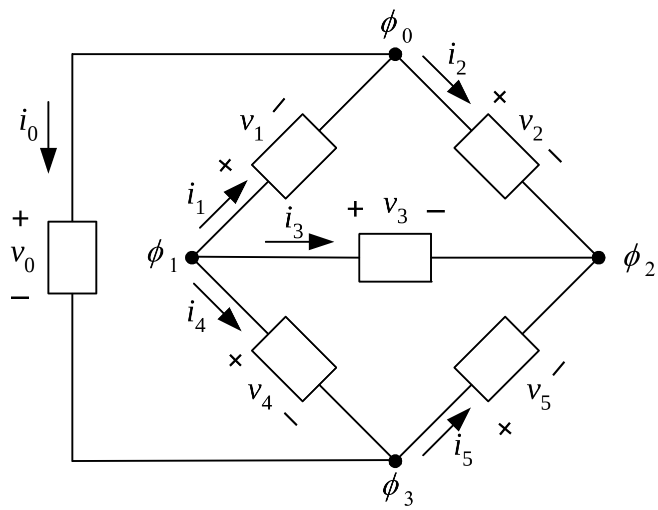
Tellegen's Theorem

Circuit:



$$v_0 = \phi_0 - \phi_3,\ \ v_1 = \phi_1 - \phi_0,\ \ v_2 = \phi_0 - \phi_2,$$
$$v_3 = \phi_3 - \phi_2,\ \ v_4 = \phi_1 - \phi_3,\ \ v_5 = \phi_3 - \phi_2.$$

Incidence Matrix:

element \ node	0	1	2	3
0	+1	0	0	–1
1	–1	+1	0	0
2	+1	0	–1	0
3	0	+1	–1	0
4	0	+1	0	–1
5	0	0	–1	+1

$= \mathbf{A}$

node \ element	0	1	2	3	4	5
0	+1	–1	+1	0	0	0
1	0	+1	0	+1	+1	0
2	0	0	–1	–1	0	–1
3	–1	0	0	0	–1	+1

$= \mathbf{A}^T$

Potential, Voltage, and Current Vectors

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \qquad \mathbf{i} = \begin{bmatrix} i_0 \\ i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}$$

Then,

$$\mathbf{v} = \mathbf{A}\boldsymbol{\phi},\ \ \mathbf{i}\cdot\mathbf{A}\boldsymbol{\phi} = \mathbf{i}\cdot\mathbf{v} = \sum_{n=0}^{5} i_n v_n\ .$$

Also, by KCL

$$\mathbf{A}^T\mathbf{i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \ \boldsymbol{\phi}\cdot\mathbf{A}^T\mathbf{i} = 0.$$

But

$$\mathbf{i}\cdot\mathbf{A}\boldsymbol{\phi} \equiv \boldsymbol{\phi}\cdot\mathbf{A}^{\mathrm{T}}\mathbf{i},$$

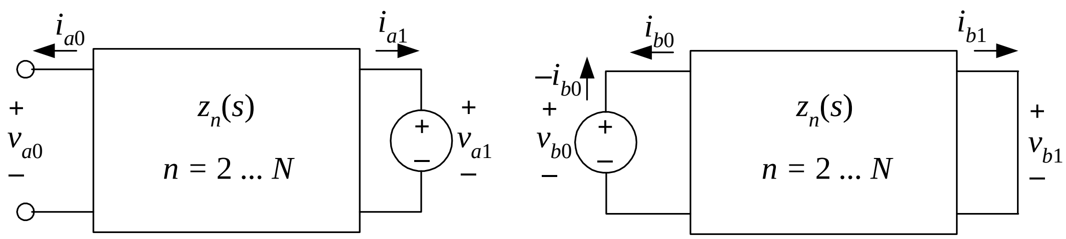
leading to Tellegen's Theorem

$$\sum_{n=0}^{5} i_n v_n = 0.$$

The only requirement is that all the i_n be for one set *a* of elements in the circuit so that KCL holds, and all the v_n be for another set *b* of elements in the circuit so that KVL holds (a set of potentials ϕ can be assigned). When set *a* is the same as set *b*, the result is simply the conservation of power. But Tellegen's Theorem is more general and leads to many other results such as reciprocity theorems. See *Tellegen's Theorem and Electrical Networks* (MIT research monograph no. 58) by Paul Penfield, Robert Spencer, S. Duinker.

Example of Using Tellegen's Theorem

Consider two networks with the same topology and, inside their respective two-port boxes, the same set of elements—passive complex impedances $z_n(s)$. The outside elements differ—an open circuit at port 0 and a source at port 1 in one case, and a source at port 0 and a short circuit at port 1 in the other case.



Choosing the currents from network *a* and the voltages from network *b* for Tellegen's Theorem,

$$0 = \sum_{n=0}^{N} i_{an} v_{bn} = i_{a0} v_{b0} + i_{a1} v_{b1} + \sum_{n=2}^{N} i_{an} v_{bn} = 0 \cdot v_{b0} + i_{a1} \cdot 0 + \sum_{n=2}^{N} i_{an} v_{bn} = \sum_{n=2}^{N} i_{an} v_{bn}$$
$$= \sum_{n=2}^{N} i_{an} i_{bn} z_n(s).$$

Choosing the voltages from network *a* and the currents from network *b* for Tellegen's Theorem,

$$0 = \sum_{n=0}^{N} v_{an} i_{bn} = v_{a0} i_{b0} + v_{a1} i_{b1} + \sum_{n=2}^{N} v_{an} i_{bn} = v_{a0} i_{b0} + v_{a1} i_{b1} + \sum_{n=2}^{N} i_{an} z_n(s) i_{bn} = v_{a0} i_{b0} + v_{a1} i_{b1} + \sum_{n=2}^{N} i_{an} i_{bn} z_n(s)$$
$$= v_{a0} i_{b0} + v_{a1} i_{b1}.$$

Therefore we have a reciprocity of the reverse open-circuit voltage transfer equaling the forward short-circuit current transfer:

$$\frac{v_{a0}}{v_{a1}} = \frac{i_{b1}}{-i_{b0}}.$$