

Text Reference: Sections 5.3 and 4.8
 Wikipedia Reference: Fibonacci Number

Lab 6: The Fibonacci Sequence and the Lucas Sequence

The purpose of this Lab is to provide an introduction to the Fibonacci sequence, which arises in number theory, applied mathematics, and biology. In the process you will see how useful eigenvalues and eigenvectors can be in understanding the dynamics of difference equations.

Fibonacci Sequence

The **Fibonacci sequence** is the sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, \dots$$

You can probably see the pattern: Each number is the sum of the two numbers immediately preceding it. Symbol y_k is the k^{th} number in the sequence (with $y_0 = 0$), called the k^{th} **Fibonacci number**.

How can y_k be found without just computing the sequence term by term? The answer to this question involves matrix multiplication and eigenvalues. The Fibonacci sequence is governed by the equations

$$y_{k+2} = y_{k+1} + y_k, \quad y_0 = 0, y_1 = 1,$$

or, equivalently,

$$y_{k+2} - y_{k+1} - y_k = 0, \quad y_0 = 0, y_1 = 1.$$

Section 4.8 in Lay's textbook 5/E identifies the last equation as a second-order linear difference equation. For reasons which will shortly become apparent, a trivial equation is added to get the following system of equations:

$$\begin{aligned} y_{k+1} &= y_{k+1} \\ y_{k+2} &= y_k + y_{k+1} \end{aligned}$$

To see how linear algebra applies to this problem, define vector $\mathbf{u}_k = \begin{bmatrix} y_k \\ y_{k+1} \end{bmatrix}$ and matrix A , where $A =$

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. The above system of equations may then be written as the vector-matrix equation $\mathbf{u}_{k+1} = A \mathbf{u}_k$.

To find Fibonacci number y_k , extract the top entry in vector \mathbf{u}_k . The vector \mathbf{u}_k could be written in terms of \mathbf{u}_0 by noting that by inductive reasoning

$$\mathbf{u}_k = A \mathbf{u}_{k-1} = A A \mathbf{u}_{k-2} = \dots = A^k \mathbf{u}_0$$

The first goal is to find an easy way to compute A^k . This is where eigenvalues and eigenvectors enter the picture.

▼ Problems to be Submitted:

▼ Problem 1.

Define $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Using Maple, compute A^5 and use it to find vector \mathbf{u}_5 and then find Fibonacci number y_5 .

▼ Problem 2.

Show that the eigenvalues of matrix A are $\lambda_1 = \frac{1 + \sqrt{5}}{2}$ and $\lambda_2 = \frac{1 - \sqrt{5}}{2}$ by solving the characteristic equation of A .

▼ Problem 3.

Show that $\begin{bmatrix} -\lambda_2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -\lambda_1 \\ 1 \end{bmatrix}$ are eigenvectors of A corresponding to λ_1 and λ_2 respectively.

You may find it helpful to note that $\lambda_1 + \lambda_2 = 1$ and that $\lambda_1 \lambda_2 = -1$.

▼ Problem 4.

A 2x2 matrix A is defined to be diagonalizable provided it has 2 eigenpairs. Explain why A is diagonalizable, by exhibiting the 2 eigenpairs.

▼ Problem 5.

A diagonalizable matrix A satisfies the equation $AP=PD$ where D is a diagonal matrix of eigenvalues and P is the augmented matrix of corresponding eigenvectors (D and P are not unique). Use the results of Problem 4 to find an invertible matrix P and a diagonal matrix D for which $A = P D P^{-1}$.

▼ Problem 6.

Use Maple to calculate D^{10} , then use it to find $A^{10} = PD^{10}P^{-1}$, vector \mathbf{u}_{10} , and Fibonacci number y_{10} . Confirm your result for y_{10} by writing out the Fibonacci sequence by hand and then consulting Wikipedia. In the solution, explain in detail why $A^{10} = PD^{10}P^{-1}$.

▼ Problem 7.

A formula for y_k may be derived using the preceding two problems. Use already derived expressions for P , D , and A^k to obtain the formula

$$A^k = \frac{1}{\sqrt{5}} \begin{bmatrix} d^k c & d^{k+1} c - c^{k+1} d \\ c^k - d^k & c^{k+1} - d^{k+1} \end{bmatrix} \quad \text{where } c = \lambda_1 = \frac{1 + \sqrt{5}}{2},$$

$$d = \lambda_2 = \frac{1 - \sqrt{5}}{2}.$$

▼ Problem 8.

Use the result of Problem 7 to find vector \mathbf{u}_k and Fibonacci number y_k . Again make note of the fact that $\text{cd} = \lambda_1 \lambda_2 = -1$. The answer should be

$$y_k = \frac{1}{\sqrt{5}} (\lambda_1^k - \lambda_2^k) = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^k - \left(\frac{1 - \sqrt{5}}{2} \right)^k \right)$$

Calculate Fibonacci number y_{10} using this formula, and compare your result to that of Problem 6.

▼ Problem 9.

Notice that the second eigenvalue λ_2 is less than 1 in absolute value, so term λ_2^k will approach zero as k approaches infinity. The following approximate equation results for the k^{th} Fibonacci number:

$$y_k \sim \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^k$$

Use this approximation to approximate $\frac{y_{k+1}}{y_k}$.

The approximation for $\frac{y_{k+1}}{y_k}$ which you found in the last question is called the **golden ratio**, or **golden**

mean. The ancient Greek mathematicians thought that this ratio was the perfect proportion for the rectangle. That the golden mean appears as the limiting ratio for the Fibonacci sequence (which occurs in nature in sunflowers, nautilus shells, and in the branching behavior of plants) makes one wonder about the interconnections between nature, beauty, and number.

Lucas Sequences

The above work on the Fibonacci sequence can be generalized to discuss any difference equation of the form

$$y_{k+2} = a y_{k+1} + b y_k$$

where a and b can be any real numbers. A sequence derived from this equation is often called a **Lucas sequence**, named for French mathematician Edouard Lucas.

▼ Problems to be Submitted:

▼ Problem 10.

Consider the Lucas sequence generated by the difference equation

$$y_{k+2} = 3 y_{k+1} - 2 y_k$$

with $y_0 = 0$ and $y_1 = 1$. Write out by hand the first seven terms of this sequence and see if you can find the pattern. Then apply eigenanalysis, as was done for the Fibonacci sequence, to find a formula for y_k .

▼ Problem 11.

Consider the Lucas sequence generated by the difference equation

$$y_{k+2} = 2 y_{k+1} - y_k$$

with $y_0 = 0$ and $y_1 = 1$. Find the pattern by writing out as many terms in the sequence as you need. Will the Fibonacci sequence analysis work in this case? Why or why not? Consult the Wikipedia page on the Lucas Sequence for a solution formula. Therein the following section applies with $a = b = S = 1$ [notation from the Wiki page]:

Repeated root [\[edit \]](#)

The case $D = 0$ occurs exactly when $P = 2S$ and $Q = S^2$ for some integer S so that $a = b = S$. In this case one easily finds that

$$U_n(P, Q) = U_n(2S, S^2) = nS^{n-1}$$

$$V_n(P, Q) = V_n(2S, S^2) = 2S^n.$$