Linear Algebra 2270-2
Due in Week 9

The ninth week finishes chapter 4 and starts the work from chapter 5. Here's the list of problems, problem notes and answers.

Section 4.4. Exercises 3, 7, 9, 13, 27

Section 4.5. Exercises 5, 7, 11, 13, 21

Section 4.6. Exercises 1, 3, 5, 7, 15, 21

Extra Credit Problem week9-1. Define a function T from $\mathcal{R}^n$ to $\mathcal{R}^m$ by the matrix multiply formula $T(\vec{x}) = A\vec{x}$. Prove that for all vectors $\vec{u}, \vec{v}$ and all constants c, (a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$, (b) $T(c\vec{u}) = cT(\vec{u})$. *Definition: T is called a* **linear transformation** *if T maps $\mathcal{R}^n$ into $\mathcal{R}^m$ and satisfies (a) and (b).*

Extra Credit Problem week9-2. Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^n$ that satisfies $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x}$. Prove that the $n \times n$ matrix A of T is orthogonal, that is, $A^TA = I$, which means the columns of A are **orthonormal**:

$$\mathbf{col}(A, i) \cdot \mathbf{col}(A, j) = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \mathbf{col}(A, i) \cdot \mathbf{col}(A, i) = 1.$$

Extra Credit Problem week9-3. Let T be a linear transformation given by $n \times n$ orthogonal matrix A. Then $\|T(\vec{x})\| = \|\vec{x}\|$ holds. Construct an example of such a matrix A for dimension $n = 3$, which corresponds to holding the z-axis fixed and rotating the xy-plane 45 degrees counter-clockwise. Draw a 3D-figure which shows the action of T on the unit cube $S = \{(x, y, z) \ : \ 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.

Problem Notes

Extra Credit Problem week9-1. Write out both sides of identities (a) and (b), replacing $T(\vec{w})$ by matrix product $A\vec{w}$ for various choices of $\vec{w}$. Then compare sides to finish the proof.

Extra Credit Problem week9-2. Equation $\|T(\vec{x})\| = \|\vec{x}\|$ means lengths are preserved by T. It also means $\|A\vec{x}\| = \|\vec{x}\|$, which applied to $\vec{x} = \mathbf{col}(I, k)$ means $\mathbf{col}(A, k)$ has length equal to $\mathbf{col}(I, k)$ (=1). Write $\|\vec{w}\|^2 = \vec{w} \cdot \vec{w} = \vec{w}^T\vec{w}$ (the latter a matrix product). Then write out the equation $\|A\vec{x}\|^2 = \|\vec{x}\|^2$, to see what you get, for various choices of unit vectors $\vec{x}$.

Extra Credit Problem week9-3. The equations for such a transformation can be written as plane rotation equations in x, y plus the identity in z. They might look like $x' = x\cos\theta - y\sin\theta$, $y' =$ similar, $z' = z$. Choose θ then test it by seeing what happens to $x = 1$, $y = 0$, $z = 0$, the answer for which is a rotation of vector $(1, 0, 0)$. The answer for A is obtained by writing the scalar equations as a matrix equation $(x', y', z')^T = A(x, y, z)^T$.