14. (Chapter 4, 7: 40 points) Let A be an $m \times n$ matrix. Denote by S_1 the row space of of A and S_2 the column space of A. It is known that S_1 and S_2 have dimension $r = \textbf{rank}(A)$. Let $\vec{p}_1, \ldots, \vec{p}_r$ be a basis for S_1 and let $\vec{q}_1, \ldots, \vec{q}_r$ be a basis for S_2. For example, select the pivot columns of A^T and A, respectively. Define $T : S_1 \to S_2$ initially by $T(\vec{p}_i) = \vec{q}_i, \quad i = 1, \ldots, r$. Extend T to all of S_1 by linearity, which means the final definition is

$$T(c_1\vec{p}_1 + \cdots + c_r\vec{p}_r) = c_1\vec{q}_1 + \cdots + c_r\vec{q}_r.$$

Prove that T is one-to-one and onto.

C+

$S_1 \perp S_2 \quad A^TA = B \Rightarrow$ square, invertible nxn matrix

$T(c_1\vec{P}_1 + \ldots + c_r\vec{P}_r) = c_1\vec{q}_1 + \cdots + c_r\vec{P}_r$

~~Proves it is one-to-one because the elements are the same after the transformation are just changed by a constant.~~

Assume $B\vec{x} = \vec{0}$, since B is invertible, this means the nullspace of B is the zero vector. There are n pivot columns in A^TA.

pivot columns of $A^T \perp$ pivot columns of A.

$\vec{P}_1 \ldots \vec{P}_r \perp \vec{q}_1 \cdots \vec{q}_r$

$A^TA\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$

$\vec{x}^TA^TA\vec{x} = \vec{0} \Rightarrow \vec{x}^T\vec{0} = \vec{0}$

A^T and A have zero vector as a nullspace

~~since the zero vector is the nullspace of A~~

The only way for T to be one-to-one and onto is if the zero vector is the nullspace of both A and A^T. Which we have proven above.

what does T have to do with the fund Thm of linear algebra? why do you think 1-1 and onto can be proved from the FTLA?

15. (Chapter 4: 20 points) Least squares can be used to find the best fit line for the points $(1,2)$, $(2,2)$, $(3,0)$. Without finding the line equation, describe how to do it, in a few sentences.

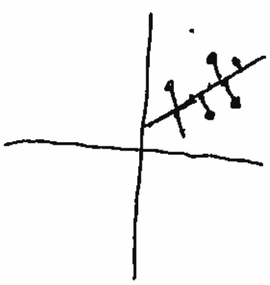
find $\vec{x}$ of $A\vec{x}=\vec{b}$ by, using $y = v_1 x + v_2$, where $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

A

Plugging that into the normal equation $A^T A \vec{y} = A^T \vec{b}$, then solve.

~~The~~ ~~[illegible]~~ The regression fits a best fit line by taking the average distance from the data points and plots a linear or non-linear line/curve. The best fit line is interpolated from the data points that have been collected.

Picture



$$y = x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

16. (Chapters 1 to 7: 20 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

B

Part 1: nullspace(A) = n − r

columnspace(A) = r

row space (A) = rm−r

nullspace(A^T) = rm−r

Part 2: nullspace(A) ⊥ nullspace(A^T) ×

nullspace(A) ⊥ row space(A)

Column space(A) ⊥ row space (A) ×

Column space(A) ⊥ nullspace(A^T)

← More revealed in #15

17. (Chapter 7: 20 points) State the **Spectral Theorem** for symmetric matrices. Include the important results included in the spectral theorem, about real eigenvalues and diagonalizability. Then discuss the **spectral decomposition.**

$A = QDQ^T$, where Q is orthogonal and D is the Diagonal Matrix, with eigenvalues on its diagonal elements.

A−

~~$A = U\Sigma V^T$~~

The Spectral Theorem States that for a real symmetric matrix, there will be, n, independent ~~eigenpai~~ eigenpairs. The matrix is then diagonalizable. For A^K, there can be multiplicity of K for eigenvalues and as long as they are independent, the ~~formula~~ formula will be $A^K = QD^KQ^T$, where the [redacted] D has the eigenvalues to the K power on its diagonal elements.

Spectral Decomp is $A^K = QD^KQ^T$
find the eigenvalues of A, these will be the diagonal elements of D, hence the diagonal matrix. Q is a matrix comprized of the eigenvectors of A, they have ~~to~~ to be orthogonal, so you can implement Q^T instead of Q^{-1}!

Not unless Gram-Schmidt was used