MATH 2270-2 Sample Exam 2 S2017

ANSWERS

Draft 30 March 2017.
No more problems added after 3 April.
Expect corrections until the exam date.

Problem 1. (5 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, find x_1, x_2 by Cramer's Rule, showing **all details** (details count 75%).

Answer:

$x_1 = \Delta_1/\Delta$, $x_2 = \Delta_2/\Delta$, $\Delta = \det\begin{pmatrix} -2 & 3 \\ 0 & -4 \end{pmatrix} = 8$, $\Delta_1 = \det\begin{pmatrix} -3 & 3 \\ 5 & -4 \end{pmatrix} = -3$, $\Delta_2 = \det\begin{pmatrix} -2 & -3 \\ 0 & 5 \end{pmatrix} = -10$, $x_1 = \frac{-3}{8}$, $x_2 = \frac{-10}{8} = \frac{-5}{4}$.

Problem 2. (5 points) Assume given 3×3 matrices A, B. Suppose $E_3E_2E_1A = BA^2$ and E_1, E_2, E_3 are elementary matrices representing respectively a multiply by 3, a swap and a combination. Assume $\det(B) = 3$. Find all possible values of $\det(-2A)$.

Answer:

Start with the determinant product theorem $|FG| = |F||G|$. Apply it to obtain $|E_3||E_2||E_1||A| = |B||A|^2$. Let $x = |A|$ in this equation and solve for x. You will need to know that $|E_1| = 3$, $|E_2| = -1$, $|E_3| = 1$. Let $C = -2A$. Then $|C| = |(-2I)A| = |-2I||A| = (-2)^3x$. The answer is $|C| = -8x$, where x is the solution of $-3x = 3x^2$. Then $|C| = 0$ or $|C| = 8$.

Problem 3. (5 points) Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Show the details of two different methods for finding A^{-1}.

Answer:

The two methods are (1) $A^{-1} = \frac{\textbf{adj}(A)}{|A|}$ and (2) For $C = < A|I >$, then $\textbf{rref}(C) = < I|A^{-1} >$. Expected details not supplied here.

Problem 4. (5 points) Find a factorization $A = LU$ into lower and upper triangular matrices for the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Answer:

Let E_1 be the result of combo(1,2,-1/2) on I, and E_2 the result of combo(2,3,-2/3) on I. Then $E_2E_1A = U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$. Let $L = E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$.

Problem 5. (5 points) Explain how the **span theorem** applies to show that the set S of all linear combinations of the functions $\cosh x, \sinh x$ is a subspace of the vector space V of all continuous functions on $-\infty < x < \infty$.

Answer:

The span theorem says $\mathbf{span}(\vec{v}_1, \vec{v}_2)$ is a subspace of V, for any two vectors in V. Choose the two vectors to be $\cosh x, \sinh x$.

Problem 6. (5 points) Write a proof that the subset S of all solutions $\vec{x}$ in $\mathcal{R}^n$ to a homogeneous matrix equation $A\vec{x} = \vec{0}$ is a subspace of $\mathcal{R}^n$. This is called the **kernel theorem**.

Answer:

(1) Zero is in S because $A\vec{0} = \vec{0}$; (2) If $A\vec{v}_1 = \vec{0}$ and $A\vec{v}_2 = \vec{0}$, then $\vec{v} = \vec{v}_1 + \vec{v}_2$ satisfies $A\vec{v} = A\vec{v}_1 + A\vec{v}_2 = \vec{0} + \vec{0} = \vec{0}$. So $\vec{v}$ is in S; (3) Let $\vec{v}_1$ be in S, that is, $A\vec{v}_1 = \vec{0}$. Let c be a constant. Define $\vec{v} = c\vec{v}_1$. Then $A\vec{v} = A(c\vec{v}_1) = cA\vec{v}_1 = (c)\vec{0} = \vec{0}$. Then $\vec{v}$ is in S. This completes the proof.

Problem 7. (5 points) Using the subspace criterion, write two hypotheses that imply that a set S in a vector space V is not a subspace of V. The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

Answer:

(1) If the zero vector is not in S, then S is not a subspace. (2) If two vectors in S fail to have their sum in S, then S is not a subspace. (3) If a vector is in S but its negative is not, then S is not a subspace.

Problem 8. (5 points) Report which columns of A are pivot columns: $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

Answer:

Zero cannot be a pivot column (no leading one in $\mathbf{rref}(A)$). The other two columns are not constant multiples of one another, therefore they are independent and will become pivot columns in $\mathbf{rref}(A)$. Then: pivot columns =2,3.

Problem 9. (5 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Answer:

The augmented matrix has reduced row echelon form (last frame) equal to the matrix $\begin{matrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$. Then $x_1 = t_1, x_2 = 1, x_3 = 1$ is the general solution in scalar form. The partial derivative on t_1 gives the homogeneous solution basis vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $\vec{x}_h = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Set $t_1 = 0$ in the scalar solution to find a particular solution $\vec{x}_p = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Problem 10. **(5 points)** Find the reduced row echelon form of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

Answer:

It is the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Problem 11. **(5 points)** A 10×13 matrix A is given and the homogeneous system $A\vec{x} = \vec{0}$ is transformed to reduced row echelon form. There are 7 lead variables. How many free variables?

Answer:

Because $\vec{x}$ has 13 variables, then the rank plus the nullity is 13. There are 6 free variables.

Problem 12. **(5 points)** The rank of a 10×13 matrix A is 7. Find the nullity of A.

Answer:

There are 13 variables. The rank plus the nullity is 13. The nullity is 6.

Problem 13. **(5 points)** Given a basis $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$ of $\mathcal{R}^2$, and $\vec{v} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$, then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ for a unique set of coefficients c_1, c_2, called the *coordinates of $\vec{v}$ relative to the basis $\vec{v}_1, \vec{v}_2$*. Compute c_1 and c_2.

Answer:

Problem 14. (5 points) Determine independence or dependence for the list of vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Answer:
Possible tests are the rank test, determinant test, pivot theorem. Let A denote the augmented matrix of the three column vectors. The determinant is 32, nonzero, so the vectors are independent. The pivot theorem also applies. The $\mathbf{rref}(A)$ is the identity matrix, so all columns are pivot columns, hence the three columns are independent. The rank test applies because the rank is 3, equal to the number of columns, hence independence.

Problem 15. (5 points) Check the independence tests which apply to prove that 1, x^2, x^3 are independent in the vector space V of all functions on $-\infty < x < \infty$.

☐	**Wronskian test**	Wronskian of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☐	**Rank test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
☐	**Determinant test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
☐	**Euler Solution Test**	Any finite set of distinct Euler solution atoms is independent.
☐	**Pivot test**	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.

Answer:
The first and fourth apply to the given functions, while the others apply only to fixed vectors.

Problem 16. (5 points) Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Prove that S is a subspace of $\mathcal{R}^3$.

Answer:
Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then the restriction equations can be written as $A\vec{x} = \vec{0}$. Apply the nullspace theorem (also called the kernel theorem), which says that the nullspace of a matrix is a subspace.

Another solution: The given restriction equations are linear homogeneous algebraic equations. Therefore, S is the nullspace of some matrix B, hence a subspace of $\mathcal{R}^3$. This solution uses the fact that linear homogeneous algebraic equations can be written as a matrix equation $B\vec{x} = \vec{0}$.

Problem 17. (5 points) The 5×6 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & -2 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 6 & 0 & 3 \\ 2 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Answer:

Find $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The pivot columns are 1 and 4.

Problem 18. (5 points) Let A be an $m \times n$ matrix with independent columns. Prove that $A^T A$ is invertible.

Answer:

The matrix $B = A^T A$ has dimension $n \times n$. We prove that the nullspace of $B = A^T A$ is the zero vector.

Let $\vec{x}$ belong to $\mathcal{R}^n$. Assume $B\vec{x} = \vec{0}$, then multiply this equation by $\vec{x}^T$ to obtain $\vec{x}^T A^T A\vec{x} = \vec{x}^T\vec{0} = 0$. Therefore, $\|A\vec{x}\|^2 = 0$, or $A\vec{x} = \vec{0}$. If A has independent columns, then the nullspace of A is the zero vector, so $\vec{x} = \vec{0}$. We have proved that the nullspace of $B = A^T A$ is the zero vector.

An $n \times n$ matrix B is invertible if and only if its nullspace is the zero vector. So $B = A^T A$ is invertible.

Problem 19. (5 points) Let A be an $m \times n$ matrix with $A^T A$ invertible. Prove that the columns of A are independent.

Answer:

The columns of A are independent if and only if the nullspace of A is the zero vector. If you don't know this result, then find it in Strang's book, or prove it yourself.

Assume $\vec{x}$ is in the nullspace of A, $A\vec{x} = \vec{0}$, then multiply by A^T to get $A^T A\vec{x} = \vec{0}$. Because $A^T A$ is invertible, then $\vec{x} = \vec{0}$, which proves the nullspace of A is the zero vector. We conclude that the columns of A are independent.

Problem 20. (5 points) Let A be an $m \times n$ matrix and $\vec{v}$ a vector orthogonal to the nullspace of A. Prove that $\vec{v}$ must be in the row space of A.

Answer:

The fundamental theorem of linear algebra is summarized by `rowspace` $\perp$ `nullspace`. This relation implies `nullspace` $\perp$ `rowspace`, because for subspaces S we have $(S^{\perp})^{\perp} = S$. The conclusion follows.

Problem 21. (5 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer:

The columns of P are the eigenvectors and the diagonal entries of D are the eigenvalues, taken in the same order.

Problem 22. (5 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer:

The characteristic polynomial is $\det(A - rI) = (-r)(3 - r)(r - 2)^2$. The eigenvalues are $0, 2, 2, 3$. Determinant expansion of $\det(A - \lambda I)$ is by the cofactor method along column 1. This reduces it to a 3×3 determinant, which can be expanded by the cofactor method along column 3.

Problem 23. (5 points) The matrix $A = \begin{pmatrix} 0 & -12 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ has eigenvalues $0, 2, 2$ but it is not diagonalizable, because $\lambda = 2$ has only one eigenpair. Find an eigenvector for $\lambda = 2$. To save time, **don't find the eigenvector for** $\lambda = 0$.

Answer:

Because $A - 2I = \begin{pmatrix} -2 & -12 & 3 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ has last frame $B = \begin{pmatrix} 1 & 0 & -15/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then there is only one eigenpair for $\lambda = 2$, with eigenvector $\vec{v} = \begin{pmatrix} 15 \\ -2 \\ 2 \end{pmatrix}$.

Problem 24. (5 points) Find the two eigenvectors corresponding to complex eigenvalues $-1 \pm 2i$ for the 2×2 matrix $A = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$.

Answer:
$\left(-1+2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(-1-2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$

Problem 25. (5 points) Let $A = \begin{pmatrix} -7 & 4 \\ -12 & 7 \end{pmatrix}$. Circle possible eigenpairs of A.

$$\left(1, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right), \quad \left(-1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right).$$

Answer:
The first and the last, because the test $A\vec{x} = \lambda\vec{x}$ passes in both cases.

Problem 26. (5 points) Let I denote the 3×3 identity matrix. Assume given two 3×3 matrices B, C, which satisfy $CP = PB$ for some invertible matrix P. Let C have eigenvalues -1, 1, 5. Find the eigenvalues of $A = 2I + 3B$.

Answer:
Both B and C have the same eigenvalues, because $\det(B - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(PCP^{-1} - \lambda PP^{-1}) = \det(C - \lambda I)$. Further, both B and C are diagonalizable. The answer is the same for all such matrices, so the computation can be done for a diagonal matrix $B = \mathbf{diag}(-1, 1, 5)$. In this case, $A = 2I + 3B = \mathbf{diag}(2, 2, 2) + \mathbf{diag}(-3, 3, 15) = \mathbf{diag}(-1, 5, 17)$ and the eigenvalues of A are $-1, 5, 17$.

Problem 27. (5 points) Let A be a 3×3 matrix with eigenpairs

$$(4, \vec{v}_1), \quad (3, \vec{v}_2), \quad (1, \vec{v}_3).$$

Let P denote the augmented matrix of the eigenvectors $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_1$, in exactly that order. Display the answer for $P^{-1}AP$. Justify the answer with a sentence.

Answer:
Because $AP = PD$, then $D = P^{-1}AP$ is the diagonal matrix of eigenvalues, taken in the order determined by the eigenpairs $(3, \vec{v}_2), (1, \vec{v}_3), (4, \vec{v}_1)$. Then $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

Problem 28. (5 points) The matrix A below has eigenvalues 3, 3 and 3. Test A to see it is diagonalizable, and if it is, then display Fourier's model for A.

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Answer:

Compute $\mathbf{rref}(A-3I) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This has rank 2, nullity 1. There is just one eigenvector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. No Fourier's model, not diagonalizable.

Problem 29. (5 points) Assume A is a given 4×4 matrix with eigenvalues $0, 1, 3 \pm 2i$. Find the eigenvalues of $4A - 3I$, where I is the identity matrix.

Answer:

Such a matrix is diagonalizable, because of four distinct eigenvalues. Then $4B - 3I$ has the same eigenvalues for all matrices B similar to A. In particular, $4A - 3I$ has the same eigenvalues as $4D - 3I$ where D is the diagonal matrix with entries $0, 1, 3+2i, 3-2i$. Compute $4D - 3I = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9+8i & 0 \\ 0 & 0 & 0 & 9-8i \end{pmatrix}$. The answer is $0, 1, 9+8i, 9-8i$.

Problem 30. (5 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -5 & 0 & 0 \\ 3 & 0 & -12 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer:

The characteristic polynomial is $\det(A - rI) = (r^2+6)(3-r)(r-2)^2$. The eigenvalues are $2, 2, 3, \pm\sqrt{6}i$. Determinant expansion is by the cofactor method along column 5. This reduces it to a 4×4 determinant, which can be expanded as a product of two quadratics. In detail, we first get $|A - rI| = (3-r)|B - rI|$, where $B = \begin{pmatrix} 0 & -2 & -5 & 0 \\ 3 & 0 & -12 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$. So we have one eigenvalue 3, and we find the eigenvalues of B. Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{c|c} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer for the eigenvalues of A is 3 and the eigenvalues of B_1 and B_3. We report $3, \pm\sqrt{6}i, 2, 2$. It is also possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$.

Problem 31. (5 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(3, \begin{pmatrix} 13 \\ 6 \\ -41 \end{pmatrix}\right), \quad \left(2i, \begin{pmatrix} i \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(-2i, \begin{pmatrix} -i \\ 2 \\ 0 \end{pmatrix}\right).$$

(1) [50%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [50%] Display a matrix product formula for A, but do not evaluate the matrix products, in order to save time.

Answer:

(1) $P = \begin{pmatrix} 13 & i & -i \\ 6 & 2 & 2 \\ -41 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}$. (2) $AP = PD$ implies $A = PDP^{-1}$.

Problem 32. (5 points) Assume two 3×3 matrices A, B have exactly the same characteristic equations. Let A have eigenvalues 2, 3, 4. Find the eigenvalues of $(1/3)B - 2I$, where I is the identity matrix.

Answer:

Because the answer is the same for all matrices similar to A (that is, all $B = PAP^{-1}$) then it suffices to answer the question for diagonal matrices. We know A is diagonalizable, because it has distinct eigenvalues. So we choose D equal to the diagonal matrix with entries $2, 3, 4$.

Compute $\frac{1}{3}D - 2I = \begin{pmatrix} \frac{2}{3} - 2 & 0 & 0 \\ 0 & \frac{3}{3} - 2 & 0 \\ 0 & 0 & \frac{4}{3} - 2 \end{pmatrix}$. Then the eigenvalues are $-\frac{4}{3}, -1, -\frac{2}{3}$.

Problem 33. (5 points) Let 3×3 matrices A and B be related by $AP = PB$ for some invertible matrix P. Prove that the roots of the characteristic equations of A and B are identical.

Answer:

The proof depends on the identity $A - rI = PBP^{-1} - rI = P(B - rI)P^{-1}$ and the determinant product theorem $|CD| = |C||D|$. We get $|A - rI| = |P||B - rI||P^{-1}| = |PP^{-1}||B - rI| = |B - rI|$. Then A and B have exactly the same characteristic equation, hence exactly the same eigenvalues.

Problem 34. (5 points) Find the eigenvalues of the matrix B:

$$B = \begin{pmatrix} 2 & 4 & -1 & 0 \\ 0 & 5 & -2 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

Answer:
The characteristic polynomial is $\det(B - rI) = (2-r)(5-r)(5-r)(3-r)$. The eigenvalues are $2, 3, 5, 5$.
It is possible to directly find the eigenvalues of B by cofactor expansion of $|B - rI|$.
An alternate method is described below, which depends upon a determinant product theorem for special block matrices, such as encountered in this example.
Matrix B is a block matrix $B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right)$, where B_1, B_2, B_3 are all 2×2 matrices. Then $B - rI = \left(\begin{array}{r|r} B_1 - rI & B_2 \\ \hline 0 & B_3 - rI \end{array}\right)$. Using the determinant product theorem for such special block matrices (zero in the left lower block) gives $|B - rI| = |B_1 - rI||B_3 - rI|$. So the answer is that B has eigenvalues equal to the eigenvalues of B_1 and B_3. These are quickly found by Sarrus' Rule applied to the two 2×2 determinants $|B_1 - rI| = (2-r)(5-r)$ and $|B_3 - rI| = r^2 - 8r + 15 = (5-r)(3-r)$.

No new questions beyond this point.Please check back at the course web site until 3 April, for corrections and added sample exam problems.