ANSWERS

Problem 1. (100 points) Define matrix $A$, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 0 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$  

For the system $A\vec{x} = \vec{b}$, find $x_3$ by Cramer’s Rule, showing all details (details count 75%). To save time, do not compute $x_1, x_2$!

Answer:

$$x_3 = \frac{\Delta_3}{\Delta}, \quad \Delta = \det \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 1 \\ 1 & 4 & 1 \end{pmatrix} = 8, \quad \Delta_3 = \det \begin{pmatrix} -2 & 3 & -3 \\ 0 & -4 & 5 \\ 1 & 4 & 1 \end{pmatrix} = 51, \quad x_3 = \frac{51}{8}.$$  

Problem 2. (100 points) Define matrix $A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$. Find a lower triangular matrix $L$ and an upper triangular matrix $U$ such that $A = LU$.

Answer:

Let $E_1$ be the result of combo(1,2,-1) on $I$, and $E_2$ the result of combo(2,3,-1) on $I$. Then

$$E_2E_1A = U = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}. \quad \text{Let} \quad L = E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$  

Problem 3. (100 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$  

Please display answers for both $\vec{x}_h$ and $\vec{x}_p$. The homogeneous solution $\vec{x}_h$ is a linear combination of Strang’s special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Answer:
The augmented matrix has reduced row echelon form (last frame) equal to the matrix
\[
\begin{pmatrix}
1 & 0 & -1/6 & 1/2 \\
0 & 1 & 1/2 & 1/2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]. Then \(x_3 = t_1, x_2 = -t_1/2 + 1/2, x_1 = t_1/6 + 1/2\) is the general solution in scalar form. The partial derivative on \(t_1\) gives the homogeneous solution basis vector \(\begin{pmatrix}
-1/6 \\
-1/2 \\
1
\end{pmatrix}\). Then \(\bar{x}_h = c_1 \begin{pmatrix}
-1/6 \\
-1/2 \\
1
\end{pmatrix}\). Set \(t_1 = 0\) in the scalar solution to find a particular solution \(\bar{x}_p = \begin{pmatrix}
1/2 \\
1/2 \\
0
\end{pmatrix}\).

**Problem 4. (100 points)** Given a basis \(\bar{v}_1 = \begin{pmatrix}
3 \\
2 \\
1
\end{pmatrix}, \bar{v}_2 = \begin{pmatrix}
4 \\
4 \\
1
\end{pmatrix}\) for a subspace \(S\) of \(\mathbb{R}^3\), and \(\bar{v} = \begin{pmatrix}
1 \\
2 \\
0
\end{pmatrix}\) in \(S\), then \(\bar{v} = c_1 \bar{v}_1 + c_2 \bar{v}_2\) for a unique set of coefficients \(c_1, c_2\), called the coordinates of \(\bar{v}\) relative to the basis \(\bar{v}_1, \bar{v}_2\). Compute \(c_1\) and \(c_2\).

Answer:
\(c_1 = -1, c_2 = 1\).

**Problem 5. (100 points)** The functions \(1, x^2, \sqrt{x^3}\) are independent in the vector space \(V\) of all functions on \(0 < x < \infty\). Check the independence tests which apply.

- **Wronskian test** Wronskian determinant of \(f_1, f_2, f_3\) nonzero at \(x = x_0\) implies independence of \(f_1, f_2, f_3\).
- **Sampling test** Sampling determinant for samples \(x = x_1, x_2, x_3\) nonzero implies independence of \(f_1, f_2, f_3\).
- **Rank test** Three vectors are independent if their augmented matrix has rank 3.
- **Determinant test** Three vectors are independent if their augmented matrix is square and has nonzero determinant.
- **Orthogonality test** Three vectors are independent if they are all nonzero and pairwise orthogonal.
- **Pivot test** Three vectors are independent if their augmented matrix \(A\) has 3 pivot columns.
The first and second apply to the given functions, while the others apply only to fixed vectors.

**Problem 6. (100 points)** Consider a $3 \times 3$ real matrix $A$ with eigenpairs

$$
\begin{pmatrix}
2, 
\begin{pmatrix}
1 \\
4 \\
-4
\end{pmatrix}, \\
1 + i, 
\begin{pmatrix}
i \\
1 \\
0
\end{pmatrix}, \\
1 - i, 
\begin{pmatrix}
-i \\
1 \\
0
\end{pmatrix}
\end{pmatrix}.
$$

(a) [60%] Display an invertible matrix $P$ and a diagonal matrix $D$ such that $AP = PD$.

(b) [40%] Display a matrix product formula for $A$. To save time, **do not evaluate** any matrix products.

**Answer:**

(a) The columns of $P$ are the eigenvectors and the diagonal entries of $D$ are the eigenvalues $2, 1 + i, 1 - i$, taken in the same order. (b) The matrix formula comes from $AP = PD$, by solving for $A = PDP^{-1}$.

**Problem 7. (100 points)** The matrix $A =
\begin{pmatrix}
3 & -1 & 0 \\
-1 & 3 & 0 \\
-1 & -1 & 4
\end{pmatrix}$ has eigenvalues 4, 4, 4. Find all eigenvectors for $\lambda = 4$. To save time, **don’t find** the eigenvector for $\lambda = 2$. Then report whether or not matrix $A$ is diagonalizable.

**Answer:**

Because $A - 4I =
\begin{pmatrix}
-1 & -1 & 0 \\
-1 & -1 & 0 \\
-1 & -1 & 0
\end{pmatrix}$ has RREF $B =
\begin{pmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}$, then there are two eigenpairs for $\lambda = 4$, with eigenvectors $\vec{v}_1 =
\begin{pmatrix}
-1 \\
1 \\
0
\end{pmatrix}$ and $\vec{v}_2 =
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}$. These answers are Strang’s special solutions for the homogeneous problem $B\vec{v} = \vec{0}$. The matrix is diagonalizable, because there are three eigenpairs.

**Problem 8. (100 points)** Using the subspace criterion, write three different hypotheses
each of which imply that a set \( S \) in a vector space \( V \) is not a subspace of \( V \). The full statement of three such hypotheses is called the **Not a Subspace Theorem**.

**Answer:**

1. If the zero vector is not in \( S \), then \( S \) is not a subspace.
2. If two vectors in \( S \) fail to have their sum in \( S \), then \( S \) is not a subspace.
3. If a vector is in \( S \) but its negative is not, then \( S \) is not a subspace.

**Problem 9. (100 points)** Define \( S \) to be the set of all vectors \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \) in \( \mathbb{R}^3 \) such that \( x_1 + x_3 = x_2, \ x_3 + x_2 = x_1 \) and \( x_1 - x_3 = 0 \). Prove that \( S \) is a subspace of \( \mathbb{R}^3 \).

**Answer:**

Let \( A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \). Then the restriction equations can be written as \( A\vec{x} = \vec{0} \).

Apply the nullspace theorem (also called the kernel theorem), which says that the nullspace of a matrix is a subspace.

Another solution: The given restriction equations are linear homogeneous algebraic equations. Therefore, \( S \) is the nullspace of some matrix \( B \), hence a subspace of \( \mathbb{R}^3 \). This solution uses the fact that linear homogeneous algebraic equations can be written as a matrix equation \( B\vec{x} = \vec{0} \).

**Problem 10. (100 points)** Let \( A \) be a 100 \( \times \) 29 matrix. Assume the columns of \( A^T A \) are independent. Prove that \( A \) has independent columns.

**Answer:**

The columns of matrix \( B \) are independent if and only if the nullspace of \( B \) is the zero vector. If you don’t know this result, then find it in Lay’s book, or prove it yourself. Also used: A square matrix has independent columns if and only if it is invertible.

Assume \( \vec{x} \) is in the nullspace of \( A \), \( A\vec{x} = \vec{0} \), then multiply by \( A^T \) to get \( A^T A\vec{x} = \vec{0} \). Because \( A^T A \) is invertible, then \( \vec{x} = \vec{0} \), which proves the nullspace of \( A \) is the zero vector. We conclude that the columns of \( A \) are independent.

**Problem 11. (100 points)** Let \( 3 \times 3 \) matrices \( A \), \( B \) and \( C \) be related by \( AP = PB \) and \( BQ = QC \) for some invertible matrices \( P \) and \( Q \). Prove that the characteristic equations of \( A \) and \( C \) are identical.
Answer:

The proof depends on the identity $A - rI = PBP^{-1} - rI = P(B - rI)P^{-1}$ and the determinant product theorem $|CD| = |C||D|$. We get $|A - rI| = |P||B - rI||P^{-1}| = |PP^{-1}||B - rI| = |B - rI|$. Then $A$ and $B$ have exactly the same characteristic equation. Similarly, $B$ and $C$ have the same characteristic equation. Therefore, $A$ and $C$ have the same characteristic equation.

**Problem 12. (100 points)** The **Fundamental Theorem of Linear Algebra** says that the null space of a matrix is orthogonal to the row space of the matrix.

Let $A$ be an $m \times n$ matrix. Define subspaces $S_1 = \text{column space of } A$, $S_2 = \text{null space of } A^T$. Prove that a vector $\vec{v}$ orthogonal to $S_2$ must be in $S_1$.

**Answer:**

The fundamental theorem of linear algebra is summarized by $\text{rowspace} \perp \text{nullspace}$. Replace $A$ by $A^T$. Then the result says that $\text{colspace}(A) \perp \text{nullspace}(A^T)$ or $S_1 \perp S_2$. Results known are $\dim \text{nullspace} = \text{nullity}$, $\dim \text{rowspace} = \text{rank}$, and $\text{rank} + \text{nullity} = \text{column dimension of the matrix}$. Because $S_1 \perp S_2$, then the two subspaces $S_1, S_2 = \text{intersect in the zero vector}$. Their dimensions are $\text{rank}(A)$ and $\text{nullity}(A^T)$. Because $\text{rank}(A) = \text{rank}(A^T)$, then their two dimensions add to $m$, as follows: $\dim(S_1) + \dim(S_2) = \text{rank}(A) + \text{nullity}(A^T) = \text{rank}(A^T) + \text{nullity}(A^T) = m$. Orthogonality of the two subspaces implies a basis for $S_1$ added to a basis for $S_2$ supplies a basis for $\mathbb{R}^m$. Vector $\vec{v}$ is given to be orthogonal to $S_2$. Then it must have a basis expansion involving only the basis for $S_1 = \text{colspace}(A)$. This proves that $\vec{v}$ is in $S_1$, which is the column space of $A$.

A **shorter proof** is possible, starting with the result that $\mathbb{R}^n$ equals the direct sum of the subspaces $S_1$ and $S_2$. The above details proved this result, as part of the solution. Then the only details left are to add the bases together and expand $\vec{v}$ in terms of the basis elements. The basis elements from $S_2$ are orthogonal to $\vec{v}$, therefore $\vec{v}$ is expressed as a linear combination of basis elements from $S_1$, which proves that $\vec{v}$ is in $S_1 = \text{the column space of } A$. 
