
Chapter 9

Eigenanalysis

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This presentation of **matrix eigenanalysis** treats the subject in depth for a 3×3 matrix A . The generalization to an $n \times n$ matrix A is to be supplied by the reader.

9.1 Eigenanalysis I

Studied here is eigenanalysis for matrix equations. The topics are *eigenanalysis*, *eigenvalue*, *eigenvector*, *eigenpair* and *diagonalization*.

What's Eigenanalysis?

The term **eigenanalysis** refers to the identification and computation of a *new coordinate system*, along with *paired scale factors*, one scale factor per coordinate direction. The new coordinate system along with the units defined by the scale factors is employed to **simplify** the expression of the original mathematical model, be it a matrix model, a differential equations model, or otherwise.

Matrix eigenanalysis applies to a matrix equation $\vec{y} = A\vec{x}$.

Differential equation eigenanalysis applies to an ordinary or partial differential equation.

Matrix Eigenanalysis

Consider the matrix equation $\vec{y} = A\vec{x}$, where symbol A is a square matrix of constants and symbols \vec{x}, \vec{y} are column vectors. The matrix

equation is equivalent to simultaneous linear algebraic equations. For 3×3 matrices, the equivalent set of linear algebraic equations is

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = y_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = y_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = y_3. \end{cases}$$

Table 1. What is Matrix Eigenanalysis?

Matrix eigenanalysis identifies a change of variables that simplifies the linear simultaneous algebraic equations represented by the system $\vec{y} = A\vec{x}$.

Eigenanalysis on a 3×3 matrix A discovers a new coordinate system $\vec{v}_1, \vec{v}_2, \vec{v}_3$ which replaces the matrix equation $\vec{y} = A\vec{x}$ with simplified equations.

Table 2. Simplification of 3×3 Linear Algebraic Equations

A change of variables $\vec{X} = P\vec{x}$, $\vec{B} = P\vec{b}$ using eigenanalysis vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ for the columns of P , simplifies a 3×3 system of linear algebraic equations $A\vec{x} = \vec{b}$ into the diagonal form

$$(1) \quad \begin{cases} \lambda_1 X_1 = B_1, \\ \lambda_2 X_2 = B_2, \\ \lambda_3 X_3 = B_3. \end{cases}$$

Scalar values $\lambda_1, \lambda_2, \lambda_3$ are scale factors (measurement units) corresponding to the directions $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Precise definitions and details appear later.

Coordinate Change via Eigenanalysis

Technically, matrix eigenanalysis is an opportunistic change of coordinates, which means the analysis must compute a set of independent column vectors that span the space. Linear algebra calls such a set of vectors a **basis**. Eigenanalysis constructs from the matrix A a *special basis*. This basis defines a change of coordinates $\vec{x} \rightarrow P\vec{x}$ where P is the augmented matrix of basis vectors.

Consider vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ which form a basis for \mathcal{R}^3 . To be a *basis* means that *each possible* vector \vec{x} in \mathcal{R}^3 can be uniquely expressed as a linear combination $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$. Geometrically, the vectors are a basis provided each possible \vec{x} in \mathcal{R}^3 can be constructed from the

triad $\vec{v}_1, \vec{v}_2, \vec{v}_3$ using solely the geometric parallelogram law for vector addition.

The change of coordinates is defined by $P = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle =$ augmented matrix of the three basis vectors. The claimed simplifying change of variables is $\vec{X} = P\vec{x}$, $\vec{Y} = P\vec{y}$, which changes $\vec{y} = A\vec{x}$ into $\vec{Y} = B\vec{X}$ where $B = PAP^{-1}$ (more details later).

Fourier's Replacement Process

J. B. Fourier (1768-1830) discovered this process in his 1822 study of heat conduction. For the case of \mathcal{R}^3 , basis vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are re-scaled by invented **scale factors** $\lambda_1, \lambda_2, \lambda_3$, which we imagine as measurement units along the three directions $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Briefly, a vector \vec{x} is replaced by a new vector \vec{y} , according to the rule

$$(2) \quad \begin{aligned} \vec{x} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \text{ implies} \\ \vec{y} &= c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + c_3\lambda_3\vec{v}_3. \end{aligned}$$

Table 3. Fourier's Re-Scaling Idea

Replace $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by re-scaled vectors $\lambda_1\vec{v}_1, \lambda_2\vec{v}_2, \lambda_3\vec{v}_3$.

Critical readers can complain that no connection has been made between the matrix A , the basis vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and the scale factors $\lambda_1, \lambda_2, \lambda_3$. For the moment, please focus on the *replacement process*, postponing the connection with A until much later, when details will be revealed.

A Key Example: Data Conversion

Let \vec{x} in \mathcal{R}^3 be a data set variable with coordinates x_1, x_2, x_3 recorded respectively in units of meters, millimeters and centimeters. Imagine the data being recorded every few milliseconds from three different sensors. The \vec{x} -data set is converted into a \vec{y} -data set with meter, kilogram, second units (MKS units) via the equations

$$(3) \quad \begin{cases} y_1 = x_1, \\ y_2 = 0.001x_2, \\ y_3 = 0.01x_3. \end{cases}$$

Equations (3) are an instance of Fourier's replacement process, Table 3. The paired scale factors and vectors are

$$\lambda_1 = 1, \quad \lambda_2 = 0.001, \quad \lambda_3 = 0.01, \\ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then equations (3) can be written as the replacement process

$$(4) \quad \begin{aligned} \vec{x} &= x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{implies} \\ \vec{y} &= x_1 \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Vectors \vec{v}_1 , \vec{v}_2 , \vec{v}_3 are the *data directions* (or *axes*) re-scaled by the measurement units λ_1 , λ_2 , λ_3 , respectively. In particular, data direction \vec{v}_2 is for millimeters.

Fourier's Replacement Process as a Matrix Equation

Theorem 1 (Matrix Form of Fourier's Replacement)

Assume vectors \vec{v}_1 , \vec{v}_2 , \vec{v}_3 are independent. Fourier's replacement equation

$$\begin{aligned} \vec{x} &= c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \quad \text{implies} \\ \vec{y} &= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + c_3 \lambda_3 \vec{v}_3 \end{aligned}$$

has the matrix equivalent

$$\vec{y} = PDP^{-1}\vec{x}, \quad \text{where } P = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Proof: The formula is justified from the scalar replacement equations by writing

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad \vec{y} = P \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \lambda_3 c_3 \end{pmatrix} = PD\vec{c}.$$

Then $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = P\vec{c}$ implies $\vec{c} = P^{-1}\vec{x}$, giving $\vec{y} = PD\vec{c} = PDP^{-1}\vec{x}$, proving the formula in Theorem 1.

The matrix $P = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle$ is the familiar augmented matrix of column vectors \vec{v}_1 , \vec{v}_2 , \vec{v}_3 . The matrix product $P\vec{c}$ expands to $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$ because matrix multiply is a linear combination of the columns. The inverse P^{-1} exists due to assumed independence of the three vectors.

Definitions: Eigenvalue, Eigenvector and Eigenpair

Eigenanalysis for the matrix equation $\vec{y} = A\vec{x}$ is the algebraic method for discovering the basis vectors \vec{v}_1 , \vec{v}_2 , \vec{v}_3 and the scale factors λ_1 , λ_2 , λ_3 . The vectors are called **eigenvectors** and the scale factors are called **eigenvalues**.

A scale factor λ is thought to be a **measurement unit** along an axis \vec{v} , therefore the eigenvectors and eigenvalues occur in pairs, called **eigenpairs**. The dependence of a pair is due to the fundamental equation below, which is used in references to define and/or compute an eigenpair.

Definition 1

An **eigenpair** (λ, \vec{v}) is defined to be a solution of the problem

$$(5) \quad A\vec{v} = \lambda\vec{v}, \quad \vec{v} \neq \vec{0}.$$

Vector \vec{v} is called an **eigenvector**. The value λ is called the **eigenvalue** corresponding to the eigenvector \vec{v} .

The motivation for the rather abstract definition of eigenpair appears below. Excuses aside, the reader is asked to know definition (5), because of its explicit use in computations.

Eigenpair computation will be delayed, in order to explain the relevance in the 3×3 case of equation (5) to the problem of computing basis vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and scale factors $\lambda_1, \lambda_2, \lambda_3$.

Important. Because $\vec{v} \neq \vec{0}$ in equation (5), then an eigenvector is never the zero vector: an eigenvector is a *direction*. Otherwise stated, an eigenvector answer of zero means you just discovered an algebra error.

History. The German term *eigenwert* was coined by David Hilbert in 1904. James J. Sylvester in 1883 coined the equivalent term **latent root**:

... the latent roots of a matrix – latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf.

By 1967, Paul Halmos gave up the battle over which words to use in his text *A Hilbert Space Problem Book*:

For many years I have battled for **proper values** (Sylvester's latent roots), and against the one and a half times translated German-English hybrid (eigenvalue) that is often used to refer to them. I have now become convinced that the war is over, and eigenvalues have won it; in this book I use them.

Eigenpair Equations and $AP = PD$

The matrix formulation $\vec{y} = PDP^{-1}\vec{x}$ reduces to vector-matrix eigenpair equations, as follows. Suppose that $\vec{y} = A\vec{x}$. Then $A\vec{x} = PDP^{-1}\vec{x}$ holds for all \vec{x} , which in turn implies that $A = PDP^{-1}$, or equivalently,

$$AP = PD.$$

Using

$$P = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

write the two matrix multiply equations AP and PD in expanded form

$$AP = \langle A\vec{v}_1 | A\vec{v}_2 | A\vec{v}_3 \rangle, \quad PD = \langle \lambda_1\vec{v}_1 | \lambda_2\vec{v}_2 | \lambda_3\vec{v}_3 \rangle.$$

Matching columns left and right in the equation $AP = PD$ then implies

$$A\vec{v}_1 = \lambda_1\vec{v}_1, \quad A\vec{v}_2 = \lambda_2\vec{v}_2, \quad A\vec{v}_3 = \lambda_3\vec{v}_3.$$

Reversibility of steps implies:

Theorem 2 (Eigenpairs and $AP = PD$)

Assume $\vec{v}_1, \vec{v}_2, \vec{v}_3$ independent. Then relations

$$(6) \quad \begin{cases} A\vec{v}_1 = \lambda_1\vec{v}_1, \\ A\vec{v}_2 = \lambda_2\vec{v}_2, \\ A\vec{v}_3 = \lambda_3\vec{v}_3. \end{cases} \quad \text{(Eigenpair Equations)}$$

hold if and only if $AP = PD$ where P and D are defined by equations

$$(7) \quad P = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Computing Eigenpairs of a Matrix

To compute an eigenpair (λ, \vec{v}) of a square matrix A requires finding scalar λ and a nonzero vector \vec{v} satisfying the homogeneous matrix-vector equation

$$A\vec{v} = \lambda\vec{v}.$$

Write it as $A\vec{x} - \lambda\vec{x} = \vec{0}$, then replace $\lambda\vec{x}$ by $\lambda I\vec{x}$ to obtain the standard linear algebraic system form¹

$$(A - \lambda I)\vec{v} = \vec{0}, \quad \vec{v} \neq \vec{0}.$$

Let $B = A - \lambda I$. The homogeneous equation $B\vec{v} = \vec{0}$ has a nonzero solution \vec{v} if and only if there are infinitely many solutions. Because the matrix is square, infinitely many solutions occur if and only if $\mathbf{rref}(B)$ has a row of zeros. Determinant theory gives a more concise statement: $\det(B) = 0$ if and only if $B\vec{v} = \vec{0}$ has infinitely many solutions. This proves:

Theorem 3 (Characteristic Equation)

The eigenvalues of a square matrix A are roots λ of the polynomial equation

$$\det(A - \lambda I) = 0.$$

¹Identity I is required to factor out the matrix $A - \lambda I$. It is wrong to factor out $A - \lambda$, because A is 3×3 and λ is 1×1 , incompatible sizes for matrix addition.

The equation is called the **characteristic equation**. The **characteristic polynomial** is the polynomial obtained by determinant evaluation on the left, using cofactor expansion or the triangular rule.

Characteristic Equation Illustration.

$$\begin{aligned} \det \left(\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) &= \begin{vmatrix} 1 - \lambda & 3 \\ 1 & 2 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(2 - \lambda) - 6 \\ &= \lambda^2 - 3\lambda - 4 \\ &= (\lambda + 1)(\lambda - 4). \end{aligned}$$

The characteristic equation $\lambda^2 - 3\lambda - 4 = 0$ has roots $\lambda_1 = -1$, $\lambda_2 = 4$. The characteristic polynomial is $\lambda^2 - 3\lambda - 4$.

Table 4. Shortcut for the Characteristic Polynomial

To find the characteristic polynomial $|A - \lambda I|$, subtract λ from the diagonal of A and then evaluate the determinant.

Theorem 4 (Finding Eigenvectors of Matrix A)

For each root λ of the characteristic equation, write the toolkit sequence for $B = A - \lambda I$ ending with $\mathbf{rref}(B)$, followed by solving for the general solution \vec{v} of the homogeneous equation $B\vec{v} = \vec{0}$, which contains invented symbols t_1, t_2, \dots . The vector basis answers $\partial_{t_1}\vec{v}, \partial_{t_2}\vec{v}, \dots$ are exactly all independent **eigenvectors** of A paired to eigenvalue λ .²

Proof: The equation $A\vec{v} = \lambda\vec{v}$ is equivalent to $B\vec{v} = \vec{0}$. Because $\det(B) = 0$, then this system has infinitely many solutions. Then a toolkit sequence starting with B and ending with $\mathbf{rref}(B)$ will have at least one row of zeros. The general solution then has one or more free variables which are assigned invented symbols t_1, t_2, \dots , and then the vector basis is obtained from the corresponding list of partial derivatives on these symbols. Each basis element is a nonzero solution of $A\vec{v} = \lambda\vec{v}$. By construction, the basis elements (eigenvectors for λ) are collectively independent. The proof is complete.

Computational Complexity. The theorem implies that a 3×3 matrix A with eigenvalues 1, 2, 3 causes three toolkit sequences to be computed, each sequence producing one eigenpair. In contrast, if A has eigenvalues 1, 1, 1, then only one toolkit sequence is computed, resulting in one, two or three eigenpairs for eigenvalue $\lambda = 1$.

²Gilbert Strang has called these vector answers **special solutions** of the homogeneous equation $B\vec{v} = \vec{0}$. Readers should justify that Strang's special solutions are independent and their linear combinations are all possible solutions of $B\vec{v} = \vec{0}$.

Fourier's Replacement for Matrices

Eigenanalysis has humble beginnings in **Fourier's Replacement Process**:

$$(8) \quad \begin{aligned} \vec{x} &= c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 \text{ implies} \\ \vec{y} &= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + c_3 \lambda_3 \vec{v}_3. \end{aligned}$$

If $\vec{y} = A\vec{x}$, where A is a 3×3 matrix, then these relations can be written as a single equation

$$(9) \quad A(c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3) = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + c_3 \lambda_3 \vec{v}_3.$$

Table 5. Fourier's Replacement for a Matrix A

To compute $A\vec{x}$ from $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$, replace each vector \vec{v}_i by its scaled version $\lambda_i \vec{v}_i$.

The replacement process depends upon the three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ being a basis for \mathcal{R}^3 , in order that $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$ represents all possible vectors in \mathcal{R}^3 .

For example, the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ fails to satisfy the replacement equation (9) for any choice of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and scalars $\lambda_1, \lambda_2, \lambda_3$. The failure is exactly due to the lack of a full set of three eigenpairs: this example has just two eigenpairs.

Fourier's Replacement Illustrated

Let

$$(10) \quad \begin{aligned} A &= \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix} \\ \lambda_1 &= 1, \quad \lambda_2 = 2, \quad \lambda_3 = -5, \\ \vec{v}_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}. \end{aligned}$$

Then Fourier's model holds (details delayed to Example 3) and

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}$$

implies

$$A\vec{x} = c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}$$

History of Fourier's Replacement

The subject of **eigenanalysis** was popularized by J. B. Fourier in his 1822 publication on the theory of heat, *Théorie analytique de la chaleur*. His ideas can be summarized as follows for the $n \times n$ matrix equation $\vec{y} = A\vec{x}$.

The vector $\vec{y} = A\vec{x}$ is obtained from eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ by replacing the eigenvectors by their scaled versions $\lambda_1\vec{v}_1, \lambda_2\vec{v}_2, \dots, \lambda_n\vec{v}_n$:

$$\begin{aligned} \vec{x} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n \text{ implies} \\ \vec{y} &= x_1\lambda_1\vec{v}_1 + x_2\lambda_2\vec{v}_2 + \dots + c_n\lambda_n\vec{v}_n. \end{aligned}$$

Powers and Fourier's Replacement

Powers A^n of a matrix A for which Fourier's Replacement equation (9) holds can be computed using only the basic vector space toolkit. To illustrate, assume 3×3 matrix A has eigenpairs (λ_i, \vec{v}_i) , $i = 1, 2, 3$. Then only the vector toolkit for \mathcal{R}^3 is used in computing

$$A^5\vec{x} = x_1\lambda_1^5\vec{v}_1 + x_2\lambda_2^5\vec{v}_2 + x_3\lambda_3^5\vec{v}_3.$$

This calculation does not use matrix multiply and it does not depend upon finding previous powers A^2, A^3, A^4 .

Fourier's replacement reduces computational effort. Matrix-vector multiplication to produce $\vec{y}_k = A^k\vec{x}$ requires $9k$ multiply operations whereas Fourier replacement gives the answer with $3k + 9$ multiply operations.

Differential Equations and Fourier's Replacement

Systems of differential equations can be solved using Fourier's replacement, giving a compact and elegant formula for the general solution. An example:

$$\begin{aligned} x_1' &= x_1 + 3x_2, \\ x_2' &= 2x_2 - x_3, \\ x_3' &= -5x_3. \end{aligned}$$

The matrix form is $\frac{d}{dt}\vec{x} = A\vec{x}$, where A and its eigenpairs are defined in equation (10).

Fourier's re-scaling idea applies to linear differential equations, as follows. First, expand the initial condition $\vec{x}(0)$ in terms of basis elements \vec{v}_1 , \vec{v}_2 , \vec{v}_3 :

$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3.$$

Then the general solution of $\frac{d}{dt}\vec{x} = A\vec{x}$ is given by replacing each \vec{v}_i by the re-scaled vector $e^{\lambda_i t}\vec{v}_i$, resulting in the formula

$$\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + c_3e^{\lambda_3 t}\vec{v}_3.$$

For the illustration here, the result is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2e^{2t} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3e^{-5t} \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

Independence of Eigenvectors

Theorem 5 (Independence of Eigenvectors)

If (λ_1, \vec{v}_1) and (λ_2, \vec{v}_2) are two eigenpairs of A and $\lambda_1 \neq \lambda_2$, then \vec{v}_1, \vec{v}_2 are linearly independent vectors.

More generally, if $(\lambda_1, \vec{v}_1), \dots, (\lambda_k, \vec{v}_k)$ are eigenpairs of A corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then $\vec{v}_1, \dots, \vec{v}_k$ are independent.

Proof: Let's solve $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ for c_1, c_2 . The vectors are independent provided the only solution is $c_1 = c_2 = 0$. Apply A to this equation, obtaining $c_1A\vec{v}_1 + c_2A\vec{v}_2 = \vec{0}$. Use $A\vec{v}_1 = \lambda_1\vec{v}_1$ and $A\vec{v}_2 = \lambda_2\vec{v}_2$ to obtain $c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$. Multiply $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ by λ_1 and subtract it from $c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$ to get $c_1(\lambda_1 - \lambda_1)\vec{v}_1 + c_2(\lambda_2 - \lambda_1)\vec{v}_2 = \vec{0}$. Because $\lambda_2 \neq \lambda_1$, cancel $\lambda_2 - \lambda_1$ to give $c_2\vec{v}_2 = \vec{0}$. The assumption $\vec{v}_2 \neq \vec{0}$ implies $c_2 = 0$. Return to the first equation and use $c_2 = 0$ to obtain $c_1\vec{v}_1 = \vec{0}$. Because $\vec{v}_1 \neq \vec{0}$, then $c_1 = 0$. This proves \vec{v}_1, \vec{v}_2 are independent.

The general case is proved by induction on k . The case $k = 1$ follows because a nonzero vector is an independent set. Assume it holds for $k - 1$ and let's prove it for k , when $k > 1$. To prove independence, solve

$$c_1\vec{v}_1 + \dots + c_k\vec{v}_k = \vec{0}$$

for c_1, \dots, c_k . Create a second equation by multiplication by A , effectively replacing each c_i by $\lambda_i c_i$ (because $A\vec{v}_i = \lambda_i\vec{v}_i$). Then multiply the first equation by λ_1 and subtract the two equations to get

$$c_1(\lambda_1 - \lambda_1)\vec{v}_1 + c_2(\lambda_1 - \lambda_2)\vec{v}_2 + \dots + c_k(\lambda_1 - \lambda_k)\vec{v}_k = \vec{0}.$$

The first term is zero. Apply the induction hypothesis to the remaining $k - 1$ vectors, then all coefficients $(\lambda_1 - \lambda_i)c_i$ are zero. Because $\lambda_1 - \lambda_i \neq 0$ for $i > 1$,

then c_2 through c_k are zero. Return to the first equation to obtain $c_1 \vec{v}_1 = \vec{0}$. Because $\vec{v}_1 \neq \vec{0}$, then $c_1 = 0$. The induction is complete.

Table 6. How to Find a Complete Set of Independent Eigenvectors

Solve the characteristic equation $|A - \lambda I| = 0$ for all eigenvalues λ . For each λ , let $B = A - \lambda I$ and solve $B\vec{v} = \vec{0}$ for general solution \vec{v} , which contains invented symbols t_1, t_2, \dots . Append the partial derivatives $\partial_{t_1} \vec{v}, \partial_{t_2} \vec{v}, \dots$ to the set of all eigenvectors for eigenvalue λ . Then the complete set of eigenvectors so obtained is independent.

Eigenanalysis Facts

1. An **eigenvalue** λ of a triangular matrix A is one of the diagonal entries. If A is non-triangular, then an eigenvalue is found as a root λ of the characteristic equation $|A - \lambda I| = 0$.
2. An **eigenvalue** of a square matrix A can be zero, positive, negative or even complex. It is a pure number, with a physical meaning inherited from the model, e.g., a scale factor or measurement unit.
3. An **eigenvector** for eigenvalue λ (a scale factor) is a nonzero direction \vec{v} of application satisfying $A\vec{v} = \lambda\vec{v}$. It is found from a toolkit sequence starting at $B = A - \lambda I$ and ending at $\text{rref}(B)$. Independent eigenvectors are computed from the general solution as partial derivatives $\partial/\partial t_1, \partial/\partial t_2, \dots$.
4. If a 3×3 matrix has three independent eigenvectors, then they collectively form a **basis** of \mathcal{R}^3 (a **coordinate system**).

Eigenanalysis and Geometry

In case the matrix A is 2×2 or 3×3 , geometry can provide additional intuition about eigenanalysis.

Fourier's 2×2 replacement $A(c_1 \vec{v}_1 + c_2 \vec{v}_2) = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2$ can be interpreted as the action of the transformation $T : \vec{x} \rightarrow A\vec{x}$ between two copies of the plane \mathcal{R}^2 ; see Figure 1.

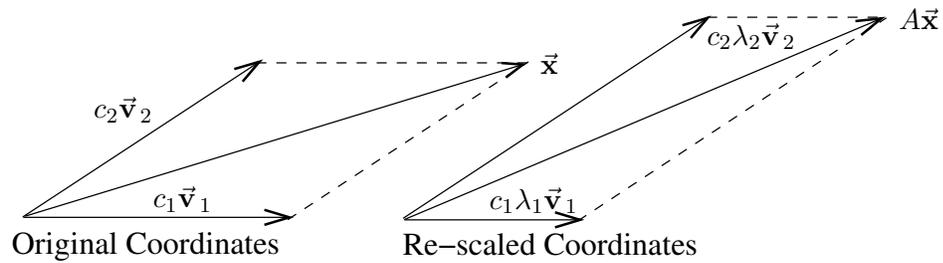


Figure 1. Transformation $T: \mathcal{R}^2 \rightarrow \mathcal{R}^2$.

Vector \vec{x} is obtained geometrically from \vec{v}_1, \vec{v}_2 by changing their lengths by c_1, c_2 , then add with the parallelogram rule. Vector $A\vec{x}$ is obtained from the two changed vectors by re-scaling by λ_1, λ_2 , then apply the parallelogram rule.

Algebraically, A is replaced by the scale factors λ_1, λ_2 and the coordinate system \vec{v}_1, \vec{v}_2 . The **eigenvalues** are the scale factors. The two vectors used in the parallelogram rule are the **eigenvectors**.

Shears are not Fourier Replacement

The important geometrical operations are scaling, shears, rotations, projections, reflections and translations. Geometrically talented readers will recognize that Fourier's replacement process describes scaling along coordinate axes.

A planar horizontal shear $(x_1, x_2) \rightarrow (y_1, y_2)$ is a set of equations

$$\begin{aligned} y_1 &= x_1 + kx_2, \quad (k = \text{shear factor} \neq 0), \\ y_2 &= x_2. \end{aligned}$$

The eigenvalues of $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ are $\lambda_1 = \lambda_2 = 1$. If it is possible to view a shear as a re-scaling, then it must be possible to change coordinates to new independent axes \vec{v}_1, \vec{v}_2 and express the shear as

$$A = PDP^{-1}, \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = \langle \vec{v}_1 | \vec{v}_2 \rangle.$$

Then $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = A = P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, a contradiction to $k \neq 0$.

Conclusion: A shear is not equivalent to scaling along axes: Fourier's replacement fails.

Diagonalization and Eigenpair Packages

All results in this subsection are valid for $n \times n$ matrices. The results are sometimes stated for the 3×3 case, to improve clarity.

Definition 2 (Diagonalizable Matrix)

An $n \times n$ matrix A which has n independent eigenvectors is called **diagonalizable**.

In practical terms, the augmented matrix $P = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle$ of eigenvectors provides a variable change $\vec{X} = P\vec{x}$, $\vec{Y} = P\vec{y}$ to transform the 3×3 system $\vec{y} = A\vec{x}$ into $\vec{Y} = D\vec{X}$, where D is a diagonal matrix:

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Theorem 6 (Diagonalization and Diagonal Matrices)

A 3×3 diagonal matrix $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ has eigenvalues on the diagonal and the corresponding eigenvectors are the columns of the 3×3 identity matrix. In summary, the eigenpairs of A are

$$\lambda_1 = a, \quad \lambda_2 = b, \quad \lambda_3 = c,$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Definition 3 (Eigenpair Packages)

Let A be a diagonalizable 3×3 matrix with eigenpairs (λ_1, \vec{v}_1) , (λ_2, \vec{v}_2) , (λ_3, \vec{v}_3) . Define the **eigenpair packages**³

$$P = \langle \vec{v}_1 | \vec{v}_2 | \vec{v}_3 \rangle, \quad D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Theorem 7 (Diagonalization)

Let A be a diagonalizable $n \times n$ matrix with eigenpair packages P, D .

1. The matrix A is completely determined by its eigenpairs:

$$A = PDP^{-1}.$$

2. The change of variables $\vec{X} = P\vec{x}$, $\vec{Y} = P\vec{y}$ transforms the equation $\vec{y} = A\vec{x}$ into the diagonal system $\vec{Y} = D\vec{X}$.

³Eigenpair packages are not unique. For 3×3 , there are six (6) permutations of the pairs, leading to six different packages. In addition, eigenvectors are not unique, leading to infinity many possible eigenpair packages.

3. The equation $A(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n$ of Fourier replacement holds with matrix form

$$AP\vec{c} = PD\vec{c}, \quad \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \text{arbitrary vector in } \mathcal{R}^n.$$

Theorem 8 (Distinct Eigenvalues)

If an $n \times n$ matrix A has n distinct eigenvalues, then it has n eigenpairs (λ_i, \vec{v}_i) , $i = 1, \dots, n$. The eigenpair packages

$$P = \langle \vec{v}_1 | \cdots | \vec{v}_n \rangle, \quad D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

satisfy $AP = PD$ and matrix A is diagonalizable.

Examples and Computational Details

1 Example (Computing 2×2 Eigenpairs)

Find all eigenpairs of the 2×2 matrix $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$.

Solution:

College Algebra. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$. Details:

$$\begin{aligned} 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\ &= \begin{vmatrix} 1 - \lambda & 0 \\ 2 & -1 - \lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\ &= (1 - \lambda)(-1 - \lambda) && \text{Sarrus' rule.} \end{aligned}$$

Linear Algebra. The eigenpairs are $\left(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, $\left(-1, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$. Details:

Eigenvector for $\lambda_1 = 1$.

$$\begin{aligned} A - \lambda_1 I &= \begin{pmatrix} 1 - \lambda_1 & 0 \\ 2 & -1 - \lambda_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 2 & -2 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} && \text{Swap and multiply rules.} \\ &= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.} \end{aligned}$$

The vector partial derivative $\partial_{t_1} \vec{v}$ of the scalar general solution $x = t_1$, $y = t_1$ is eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = -1$.

$$\begin{aligned}
A - \lambda_2 I &= \begin{pmatrix} 1 - \lambda_2 & 0 \\ 2 & -1 - \lambda_2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \\
&\approx \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\
&= \mathbf{rref}(A - \lambda_2 I) && \text{Reduced echelon form.}
\end{aligned}$$

The vector partial derivative $\partial_{t_1} \vec{v}$ of the scalar general solution $x = 0, y = t_1$ is eigenvector $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2 Example (Computing 2×2 Complex Eigenpairs)

Find all eigenpairs of the 2×2 matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

Solution:

College Algebra. The eigenvalues are $\lambda_1 = 1 + 2i, \lambda_2 = 1 - 2i$. Details:

$$\begin{aligned}
0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\
&= \begin{vmatrix} 1 - \lambda & 2 \\ -2 & 1 - \lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\
&= (1 - \lambda)^2 + 4 && \text{Sarrus' rule.}
\end{aligned}$$

The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1 - \lambda)^2 + 4 = 0$. Alternatively, use $(1 - \lambda)^2 = -4$ and take square roots.

Linear Algebra. The eigenpairs are $\left(1 + 2i, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right), \left(1 - 2i, \begin{pmatrix} i \\ 1 \end{pmatrix}\right)$.

Eigenvector for $\lambda_1 = 1 + 2i$.

$$\begin{aligned}
A - \lambda_1 I &= \begin{pmatrix} 1 - \lambda_1 & 2 \\ -2 & 1 - \lambda_1 \end{pmatrix} \\
&= \begin{pmatrix} -2i & 2 \\ -2 & -2i \end{pmatrix} \\
&\approx \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 0 & 0 \\ 1 & i \end{pmatrix} && \text{Combination rule, multiplier} = -i. \\
&\approx \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix} && \text{Swap rule.} \\
&= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.}
\end{aligned}$$

The partial derivative $\partial_{t_1} \vec{v}$ of the general solution $x = -it_1, y = t_1$ is eigenvector $\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 1 - 2i$.

The problem $(A - \lambda_2 I)\vec{v} = \vec{0}$ has solution $\vec{v} = \overline{\vec{v}_1}$. To see why, takes conjugates⁴ across the equation to get $(A - \overline{\lambda_2} I)\vec{v} = \vec{0}$. Then $\lambda_1 = \overline{\lambda_2}$ gives $\vec{v} = \overline{\vec{v}_1} = \begin{pmatrix} i \\ 1 \end{pmatrix}$.

3 Example (Computing 3×3 Eigenpairs: Real Eigenvalues)

Find all eigenpairs of the 3×3 matrix

$$(11) \quad A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix}.$$

Solution:

The answers are

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -5,$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

College Algebra. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -5$, because matrix A is triangular and the eigenvalues of a triangular matrix appear on the diagonal.

Linear Algebra. There are three toolkit sequences to compute, one for each distinct eigenvalue. The three computations appear below.

Eigenvector for $\lambda_1 = 1$.

Subtract $\lambda_1 = 1$ from the diagonal of A to obtain the equation $B\vec{v} = \vec{0}$, where

$$B = A - \lambda_1 I = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -6 \end{pmatrix}.$$

A toolkit sequence with swap, combo, multiply will find

$$\mathbf{rref}(B) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The lead variables are v_2, v_3 and the free variable is v_1 . Assign invented symbol t_1 to the free variable and back-substitute into $B\vec{v} = \vec{0}$ to obtain the scalar equations

$$\begin{aligned} v_1 &= t_1, \\ v_2 &= 0, \\ v_3 &= 0. \end{aligned}$$

⁴Overbar \bar{z} is used to indicate the complex conjugate of z , which for vectors is formally obtained componentwise by replacing complex unit i by $-i$ (reflection through the x -axis). For instance, $\overline{2 - 3i}$ equals $2 + 3i$ (replace i by $-i$).

Take the partial derivative on invented symbol t_1 across these equations to obtain the eigenvector

$$\vec{v}_1 = \begin{pmatrix} \frac{\partial v_1}{\partial t_1} \\ \frac{\partial v_2}{\partial t_1} \\ \frac{\partial v_3}{\partial t_1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = 2$.

Subtract $\lambda_2 = 2$ from the diagonal of A to obtain the equation $B\vec{v} = \vec{0}$, where

$$B = A - \lambda_2 I = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -7 \end{pmatrix}.$$

A toolkit sequence finds

$$\text{rref}(B) = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The lead variables are v_1, v_3 and the free variable is v_2 . Assign invented symbol t_1 to the free variable and back-substitute into $B\vec{v} = \vec{0}$ to obtain the scalar equations

$$\begin{aligned} v_1 &= 3t_1, \\ v_2 &= t_1, \\ v_3 &= 0. \end{aligned}$$

Take the partial derivative on invented symbol t_1 across these equations to obtain the eigenvector

$$\vec{v}_2 = \begin{pmatrix} \frac{\partial v_1}{\partial t_1} \\ \frac{\partial v_2}{\partial t_1} \\ \frac{\partial v_3}{\partial t_1} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

The eigenpair is $(\lambda_2, \vec{v}_2) = \left(2, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right)$

Eigenvector for $\lambda_3 = -5$.

Subtract $\lambda_3 = -5$ from the diagonal of A to obtain the equation $B\vec{v} = \vec{0}$, where

$$B = A - \lambda_3 I = \begin{pmatrix} 6 & 3 & 0 \\ 0 & 7 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A toolkit sequence finds

$$\text{rref}(B) = \begin{pmatrix} 1 & 0 & 1/14 \\ 0 & 1 & -1/7 \\ 0 & 0 & 0 \end{pmatrix}.$$

The lead variables are v_1, v_2 and the free variable is v_3 . Assign invented symbol t_1 to the free variable and back-substitute into $B\vec{v} = \vec{0}$ to obtain the scalar

equations

$$\begin{aligned}v_1 &= -\frac{1}{14}t_1, \\v_2 &= \frac{1}{7}t_1, \\v_3 &= 0.\end{aligned}$$

Take the partial derivative on invented symbol t_1 across these equations to obtain the eigenvector

$$\vec{v}_3 = \begin{pmatrix} \frac{\partial v_1}{\partial t_1} \\ \frac{\partial v_2}{\partial t_1} \\ \frac{\partial v_3}{\partial t_1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{14} \\ \frac{1}{7} \\ 1 \end{pmatrix}.$$

It is usual when encountering fractions in an eigenvector to replace the answer \vec{v} by $c\vec{v}$ where $c \neq 0$ is chosen to make the answer fraction-free and the first nonzero entry positive. In this case, $c = -14$ is used, and we replace \vec{v}_3 by $-14\vec{v}_3$. The eigenpair is $(\lambda_3, \vec{v}_3) = \left(-5, \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}\right)$.

This completes the computation of all three eigenpairs.

Fourier's Replacement. The equation is

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}$$

implies

$$A\vec{x} = c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}$$

Answer Check. The eigenpair equations are equivalent to the matrix identity $AP = PD$ where P is the matrix of eigenvectors and D is the diagonal matrix of corresponding eigenvalues:

$$P = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -14 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

Eigenpairs are checked by expanding AP and PD , then compare for equality. In this case, the correctness of the eigenpairs is checked by the result

$$AP = \begin{pmatrix} 1 & 6 & -5 \\ 0 & 2 & 10 \\ 0 & 0 & 70 \end{pmatrix} = PD.$$

4 Example (Computing 3×3 Eigenpairs: Complex Eigenvalues)

Find all eigenpairs of the 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Solution:

College Algebra. The eigenvalues are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$, $\lambda_3 = 3$.
Details:

$$\begin{aligned}
0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\
&= \begin{vmatrix} 1 - \lambda & 2 & 0 \\ -2 & 1 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{vmatrix} && \text{Subtract } \lambda \text{ from the diagonal.} \\
&= ((1 - \lambda)^2 + 4)(3 - \lambda) && \text{Cofactor rule and Sarrus' rule.}
\end{aligned}$$

Root $\lambda = 3$ is found from the factored form above. The roots $\lambda = 1 \pm 2i$ are found from the quadratic formula after expanding $(1 - \lambda)^2 + 4 = 0$. Alternatively, take roots across $(\lambda - 1)^2 = -4$.

Linear Algebra.

The eigenpairs are $\left(1 + 2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right)$, $\left(1 - 2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$.

Eigenvector for $\lambda_1 = 1 + 2i$.

$$\begin{aligned}
A - \lambda_1 I &= \begin{pmatrix} 1 - \lambda_1 & 2 & 0 \\ -2 & 1 - \lambda_1 & 0 \\ 0 & 0 & 3 - \lambda_1 \end{pmatrix} \\
&= \begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \\ 0 & 0 & 2 - 2i \end{pmatrix} && \text{Subtract } \lambda_1 = 1 + 2i \text{ from the diagonal.} \\
&\approx \begin{pmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Multiply rule.} \\
&\approx \begin{pmatrix} 0 & 0 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{Combination rule, factor} = -i. \\
&\approx \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} && \text{Swap rule.} \\
&= \text{rref}(A - \lambda_1 I) && \text{Reduced echelon form.}
\end{aligned}$$

The vector partial derivative $\partial_{t_1} \vec{v}$ of the scalar general solution $x = -it_1$, $y = t_1$, $z = 0$ is eigenvector $\vec{v}_1 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 1 - 2i$.

There is one eigenpair $(1 - 2i, \vec{v}_2)$, where $\vec{v}_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

Details. To see why, take conjugates⁵ across the equation $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$ to give $(\overline{A} - \overline{\lambda_2} I)\overline{\vec{v}_2} = \vec{0}$. Then $\overline{A} = A$ (A is real) and $\lambda_1 = \overline{\lambda_2}$ gives $(A - \lambda_1 I)\overline{\vec{v}_2} = \vec{0}$. Then $\overline{\overline{\vec{v}_2}} = \vec{v}_1$. Finally, $\vec{v}_2 = \overline{\overline{\vec{v}_2}} = \overline{\vec{v}_1} = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

⁵The complex conjugate is defined by $\overline{a + ib} = a - ib$ (replace i by $-i$). Two useful rules are $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ and $\overline{z_1 z_2} = \overline{z_1} \overline{z_2}$. Conjugation rules extend to vectors and matrices by applying scalar rules componentwise, e.g., $\overline{\vec{u} + \vec{v}} = \overline{\vec{u}} + \overline{\vec{v}}$ and $\overline{A\vec{x}} = \overline{A} \overline{\vec{x}}$.

Eigenvector for $\lambda_3 = 3$.

$$\begin{aligned}
 A - \lambda_3 I &= \begin{pmatrix} 1 - \lambda_3 & 2 & 0 \\ -2 & 1 - \lambda_3 & 0 \\ 0 & 0 & 3 - \lambda_3 \end{pmatrix} \\
 &= \begin{pmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 &\approx \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Multiply rule.} \\
 &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply.} \\
 &= \mathbf{rref}(A - \lambda_3 I) && \text{Reduced echelon form.}
 \end{aligned}$$

The partial derivative $\partial_{t_1} \vec{v}$ of the general solution $x = 0$, $y = 0$, $z = t_1$ is eigenvector $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

5 Example (Data Conversion) The data conversion problem

$$\begin{cases} y_1 = x_1, \\ y_2 = 0.001x_2, \\ y_3 = 0.01x_3. \end{cases}$$

is diagonalizable. The three eigenpairs of A are defined by

$$\lambda_1 = 1, \quad \lambda_2 = 0.001, \quad \lambda_3 = 0.01, \\
 \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Solution: The example was introduced above in equation (3). These equations can be written as $\vec{y} = A\vec{x}$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & 10^{-2} \end{pmatrix}$ is already a diagonal matrix.

The reader is expected to apply supporting theorems to find the eigenpairs. For practise, the eigenpairs of A can be found as in the previous example, an exercise which can discover the proof of the basic results about eigenpairs of diagonal matrices.

The answers should be verified directly from the eigenpair equation $A\vec{v} = \lambda\vec{v}$.

For instance, when $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\lambda = 0.001$, then the two sides $A\vec{v}$ and $\lambda\vec{v}$

are computed from matrix multiply, each giving the same answer $\begin{pmatrix} 0 \\ 0.001 \\ 0 \end{pmatrix}$, therefore $A\vec{v} = \lambda\vec{v}$ is valid and (λ, \vec{v}) is an eigenpair of A .

6 Example (Decomposition $A = PDP^{-1}$)

Decompose $A = PDP^{-1}$ where P, D are eigenvector and eigenvalue packages, respectively, for the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Write explicitly Fourier's replacement equation in vector-matrix notation.

Solution: By the preceding example, the eigenpairs are

$$\left(1 + 2i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(1 - 2i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

The packages are therefore

$$D = \mathbf{diag}(1 + 2i, 1 - 2i, 3), \quad P = \begin{pmatrix} -i & i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Fourier's replacement equation. The action of A in the model

$$A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + c_3\lambda_3\vec{v}_3$$

is to replace the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by scaled vectors $\lambda_1\vec{v}_1, \lambda_2\vec{v}_2, \lambda_3\vec{v}_3$. In vector form, the model is

$$AP\vec{c} = PD\vec{c}, \quad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Then the action of A is to replace eigenvector package P by the re-scaled package PD . Explicitly,

$$\begin{aligned} \vec{x} &= c_1 \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ implies} \\ A\vec{x} &= c_1(1 + 2i) \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + c_2(1 - 2i) \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + c_3(3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

7 Example (Diagonalization I)

Report **diagonalizable** or **non-diagonalizable** for the 4×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

If A is diagonalizable, then assemble and report eigenvector and eigenvalue packages P, D .

Solution: The matrix A is **non-diagonalizable**, because it fails to have 4 eigenpairs. The details:

Eigenvalues.

$$\begin{aligned}
 0 &= \det(A - \lambda I) && \text{Characteristic equation.} \\
 &= \begin{vmatrix} 1 - \lambda & 2 & 0 & 0 \\ -2 & 1 - \lambda & 0 & 0 \\ 0 & 0 & 3 - \lambda & 1 \\ 0 & 0 & 0 & 3 - \lambda \end{vmatrix} \\
 &= \begin{vmatrix} 1 - \lambda & 2 \\ -2 & 1 - \lambda \end{vmatrix} (3 - \lambda)^2 && \text{Cofactor expansion applied twice.} \\
 &= ((1 - \lambda)^2 + 4)(3 - \lambda)^2 && \text{Sarrus' rule.}
 \end{aligned}$$

The roots are $1 \pm 2i$, 3 , 3 , listed according to multiplicity.

Eigenpairs. They are

$$\left(1 + 2i, \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix} \right), \quad \left(1 - 2i, \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} \right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right).$$

Because only three eigenpairs exist, instead of four, then the matrix A is **non-diagonalizable**. Details:

Eigenvector for $\lambda_1 = 1 + 2i$.

$$\begin{aligned}
 A - \lambda_1 I &= \begin{pmatrix} 1 - \lambda_1 & 2 & 0 & 0 \\ -2 & 1 - \lambda_1 & 0 & 0 \\ 0 & 0 & 3 - \lambda_1 & 1 \\ 0 & 0 & 0 & 3 - \lambda_1 \end{pmatrix} \\
 &= \begin{pmatrix} -2i & 2 & 0 & 0 \\ -2 & -2i & 0 & 0 \\ 0 & 0 & 2 - 2i & 1 \\ 0 & 0 & 0 & 2 - 2i \end{pmatrix} \\
 &\approx \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 2 - 2i & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Multiply rule, three times.} \\
 &\approx \begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Combination and multiply rule.} \\
 &\approx \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{Combination and multiply rule.} \\
 &= \mathbf{rref}(A - \lambda_1 I) && \text{Reduced echelon form.}
 \end{aligned}$$

The general solution is $x_1 = -it_1$, $x_2 = t_1$, $x_3 = 0$, $x_4 = 0$. Then ∂_{t_1} applied to this solution gives the reported eigenpair for $\lambda = 1 + 2i$.

Eigenvector for $\lambda_2 = 1 - 2i$.

Because λ_2 is the conjugate of λ_1 and A is real, then an eigenpair for λ_2 is found by taking the complex conjugate of the eigenpair reported for λ_1 .

Eigenvector for $\lambda_3 = 3$. In theory, there can be one or two eigenpairs to report. It turns out there is only one, because of the following details.

$$\begin{aligned}
 A - \lambda_3 I &= \begin{pmatrix} 1 - \lambda_3 & 2 & 0 & 0 \\ -2 & 1 - \lambda_3 & 0 & 0 \\ 0 & 0 & 3 - \lambda_3 & 1 \\ 0 & 0 & 0 & 3 - \lambda_3 \end{pmatrix} \\
 &= \begin{pmatrix} -2 & 2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
 &\approx \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && \text{Multiply rule, two times.} \\
 &\approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} && \text{Combination and multiply rule.} \\
 &= \mathbf{rref}(A - \lambda_3 I) && \text{Reduced echelon form.}
 \end{aligned}$$

Apply ∂_{t_1} to the general solution $x_1 = 0$, $x_2 = 0$, $x_3 = t_1$, $x_4 = 0$ to give the eigenvector matching the eigenpair reported above for $\lambda = 3$.

8 Example (Diagonalization II)

Report **diagonalizable** or **non-diagonalizable** for the 4×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

If A is diagonalizable, then assemble and report eigenvalue and eigenvector packages D , P .

Solution: The matrix A is **diagonalizable**, because it has 4 eigenpairs

$$\left(1 + 2i, \begin{pmatrix} -i \\ 1 \\ 0 \\ 0 \end{pmatrix} \right), \quad \left(1 - 2i, \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} \right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right), \quad \left(3, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

Then the eigenpair packages are given by

$$D = \begin{pmatrix} -1 + 2i & 0 & 0 & 0 \\ 0 & 1 - 2i & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} -i & i & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The details parallel the previous example, except for the calculation of eigenvectors for $\lambda_3 = 3$. In this case, the reduced echelon form of $A - \lambda_3 I$ has two rows of zeros and parameters t_1, t_2 appear in the general solution. The answers given above for eigenvectors correspond to the partial derivatives $\partial_{t_1}, \partial_{t_2}$ applied to the general solution of $(A - 3I)\vec{x} = \vec{0}$.

9 Example (Non-diagonalizable Matrices)

Verify that the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are all non-diagonalizable.

Solution: Let A denote any one of these matrices and let n be its dimension.

First, we will decide on diagonalization, without computing eigenpairs. Assume, in order to reach a contradiction, that eigenpair packages D, P exist with D diagonal and P invertible such that $AP = PD$. Because A is triangular, its eigenvalues appear already on the diagonal of A . Only 0 is an eigenvalue and its multiplicity is n . Then the package D of eigenvalues is the zero matrix and an equation $AP = PD$ reduces to $AP = 0$. Multiply $AP = 0$ on the right by P^{-1} to obtain $A = 0$. But A is not the zero matrix, a contradiction. We conclude that A is not diagonalizable.

Second, we attack the diagonalization question directly, by solving for the eigenvectors corresponding to $\lambda = 0$. The toolkit sequence starts with $B = A - \lambda I$, but B equals $\mathbf{rref}(B)$ and no computations are required. The resulting reduced echelon system is just $x_1 = 0$, giving $n - 1$ free variables. Therefore, the eigenvectors of A corresponding to $\lambda = 0$ are the last $n - 1$ columns of the identity matrix I . Because A does not have n independent eigenvectors, then A is not diagonalizable.

Similar examples of non-diagonalizable matrices A can be constructed with A having from 1 up to $n - 1$ independent eigenvectors. The examples with ones on the super-diagonal and zeros elsewhere have exactly one eigenvector.

10 Example (Fourier's 1822 Heat Model)

Fourier's 1822 treatise *Théorie analytique de la chaleur* studied dissipation of heat from a laterally insulated welding rod with ends held at 0°C (ice-packed ends). Assume the initial heat distribution along the rod at time $t = 0$ is given as a linear combination

$$f = c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3.$$

Symbols $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are in the vector space V of all twice continuously differentiable functions on $0 \leq x \leq 1$, given explicitly as

$$\vec{v}_1 = \sin \pi x, \quad \vec{v}_2 = \sin 2\pi x, \quad \vec{v}_3 = \sin 3\pi x.$$

Fourier's heat model re-scales⁶ each of these vectors to obtain the temperature $u(t, x)$ at position x along the rod and time $t > 0$ as the model equation

$$u(t, x) = c_1 e^{-\pi^2 t} \vec{v}_1 + c_2 e^{-4\pi^2 t} \vec{v}_2 + c_3 e^{-9\pi^2 t} \vec{v}_3.$$

Verify that $u(t, x)$ solves Fourier's partial differential equation heat model

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \\ u(0, x) &= f(x), \quad 0 \leq x \leq 1, \quad \text{initial temperature,} \\ u(t, 0) &= 0, \quad \text{zero Celsius at rod's left end,} \\ u(t, 1) &= 0, \quad \text{zero Celsius at rod's right end.} \end{aligned}$$

Solution: First, we prove that the partial differential equation is satisfied by Fourier's solution $u(t, x)$. This is done by expanding the left side (LHS) and right side (RHS) of the differential equation, separately, then comparing the answers for equality.

Trigonometric functions $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are solutions of three different linear ordinary differential equations: $u'' + \pi^2 u = 0$, $u'' + 4\pi^2 u = 0$, $u'' + 9\pi^2 u = 0$. Because of these differential equations, we can compute directly

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 c_1 e^{-\pi^2 t} \vec{v}_1 - 4\pi^2 c_2 e^{-4\pi^2 t} \vec{v}_2 - 9\pi^2 c_3 e^{-9\pi^2 t} \vec{v}_3.$$

Similarly, computing $\partial_t u(t, x)$ involves just the differentiation of exponential functions, giving

$$\frac{\partial u}{\partial t} = -\pi^2 c_1 e^{-\pi^2 t} \vec{v}_1 - 4\pi^2 c_2 e^{-4\pi^2 t} \vec{v}_2 - 9\pi^2 c_3 e^{-9\pi^2 t} \vec{v}_3.$$

Because the second display is exactly the first, then LHS = RHS, proving that the partial differential equation is satisfied.

The relation $u(0, x) = f(x)$ is proved by observing that each exponential factor becomes $e^0 = 1$ when $t = 0$.

The two relations $u(t, 0) = u(t, 1) = 0$ hold because each of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ vanish at $x = 0$ and $x = 1$. The verification is complete.

The exercises are in progress, mostly incomplete. August 2016. Exercises 9.1

Eigenanalysis. Classify as true or false. If false, then correct the text to make it true.

1. The purpose of eigenanalysis is to discover a new coordinate system.

2. Eigenanalysis can discover an opportunistic change of coordinates.

3. Diagonal matrices have all their eigenvalues on the last row.

4. Eigenvalues are scale factors, imag-

⁶The scale factors are not constants nor are they eigenvalues, but rather, they are exponential functions of t , as was the case for matrix differential equations $\vec{x}' = A\vec{x}$

ined to be measurement units.

5. For each eigenvalue of a matrix A , there always exists at least one eigenpair.
6. Eigenvectors are independent directions.
7. Eigenvalues of a square matrix cannot be zero.
8. Eigenvectors of a square matrix cannot be the zero vector.
9. Eigenpairs (λ, \vec{v}) of a square matrix A satisfy the homogeneous equation $(A - \lambda I)\vec{v} = \vec{0}$.

Eigenpairs of a Diagonal Matrix.
Find the eigenpairs of A .

10. $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

11. $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

12. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

13. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

14. $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$

15. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Fourier's Replacement Process.
Let symbols c_1, c_2 represent arbitrary constants. Let 2×2 matrix A have Fourier replacement equation

$$A \left(c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = 2c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 5c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

16. Display the eigenpairs of A .

17. Display the replacement equation if the eigenvalues $1, -5$ are replaced by $1, 0$.

18. Display the eigenpair packages P, D such that $AP = PD$.

19. Find A .

Eigenanalysis Facts. Mark as true or false, then explain your answer.

20. If matrix A has all eigenvalues zero, then A is the zero matrix.

21. If 2×2 matrix A has all eigenvalues zero, then Fourier's replacement equation is

$$A(c_1 \vec{v}_1 + c_2 \vec{v}_2) = \vec{0}.$$

22. There are infinitely many 2×2 matrices A with complex eigenvalues $1 + i, 1 - i$.

23. A real 2×2 matrix A with eigenvalues $2 + 3i, 2 - 3i$ cannot have a real eigenvector.

24. A real 2×2 matrix A with real eigenvalues has only real eigenvectors.

Eigenpair Packages and $AP = PD$.

25. Suppose $AP = PD$ holds. Explain why there are so many different answers for P, D .

26. Suppose $AP = PD$ and $AQ = QD$ hold (same diagonal matrix D). Does $P = Q$?

27. Find one choice of P and D for $A = 2 \times 2$ diagonal matrix.

28. Find one choice of P and D for $A = 3 \times 3$ zero matrix, with column 1 of P equal to $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Matrix Eigenanalysis Method.

29. The eigenvalues of $\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ satisfy a quadratic equation. Find the equation and solve for the eigenvalues.

30. Find the eigenvalues of $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

31. A triangular $n \times n$ matrix has n eigenpairs. Provide a proof.

32. Find all eigenpairs of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

33. Prove that equations $A\vec{x} = \lambda\vec{x}$ and $(A - \lambda I)\vec{x} = \vec{0}$ have exactly the same solutions \vec{x} .

34. Cite basic linear algebra theorems to prove that $(A - \lambda I)\vec{x} = \vec{0}$ has a nonzero solution \vec{x} if and only if $|A - \lambda I| = 0$.

Basis of Eigenvectors.

35.

Independence of Eigenvectors.

36.

Diagonalization Theory.

37.

Non-diagonalizable Matrices.

38.

Distinct Eigenvalues.

39. $\begin{pmatrix} 2 & 6 \\ 5 & 3 \end{pmatrix}$

40. $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

41. $\begin{pmatrix} 2 & 6 & 2 \\ 9 & 3 & 9 \\ 1 & 3 & 1 \end{pmatrix}$

42. $\begin{pmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 3 \end{pmatrix}$

43. $\begin{pmatrix} 7 & 12 & 6 \\ 2 & 2 & 2 \\ -7 & -12 & -6 \end{pmatrix}$

44. $\begin{pmatrix} 2 & 2 & -6 \\ -3 & -4 & 3 \\ -3 & -4 & -1 \end{pmatrix}$

Computing 2×2 Eigenpairs.

45.

Computing 2×2 Complex Eigenpairs.

46.

Computing 3×3 Eigenpairs.

47. Suppose A is row-reduced to a triangular form B . Are the eigenvalues of B also the eigenvalues of A ? Give a proof or a counter-example.

48.

49.

Decomposition $A = PDP^{-1}$.

50.

Diagonalization I

51.

Diagonalization II

52.

Non-diagonalizable Matrices

53. Invent a non-diagonalizable 3×3 matrix which has exactly one eigenpair.

Fourier's Heat Model

54.