Elementary Matrices and Toolkit Sequences

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Elementary Matrices

Definition. An elementary matrix E is the result of applying a combination, multiply or swap rule to the identity matrix.

An elementary matrix is then the **second toolkit frame** after a `combo`, `swap` or `mult` toolkit operation which has been applied to a **first frame** equal to the identity matrix.

Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ First frame = identity matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{pmatrix}$$ Second frame
Elementary combo matrix
`combo(1,3,-5)`

Computer algebra systems and elementary matrices

The computer algebra system `maple` displays typical 4×4 elementary matrices (C=Combination, M=Multiply, S=Swap) as follows.

```
with(linalg):                with(LinearAlgebra):
Id:=diag(1,1,1,1);           Id:=IdentityMatrix(4);
C:=addrow(Id,2,3,c);         C:=RowOperation(Id,[3,2],c);
M:=mulrow(Id,3,m);           M:=RowOperation(Id,3,m);
S:=swaprow(Id,1,4);          S:=RowOperation(Id,[4,1]);
```

The answers:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Constructing elementary matrices E and their inverses E^{-1} _____

    Mult   Change a one in the identity matrix to symbol $m \neq 0$.

    Combo   Change a zero in the identity matrix to symbol c.

    Swap   Interchange two rows of the identity matrix.

Constructing E^{-1} from elementary matrix E

    Mult   Change diagonal multiplier $m \neq 0$ in E to $1/m$.

    Combo   Change multiplier c in E to $-c$.

    Swap   The inverse of E is E itself.

Fundamental Theorem on Elementary Matrices

Theorem 1 (Toolkit sequences and elementary matrices)
In a toolkit sequence, let the second frame A_2 be obtained from the first frame A_1 by a `combo`, `swap` or `mult` toolkit operation. Let n equal the row dimenson of A_1. Then there is correspondingly an $n \times n$ `combo`, `swap` or `mult` elementary matrix E such that

$$A_2 = EA_1.$$

Theorem 2 (The `rref` and elementary matrices)
Let A be a given matrix of row dimension n. Then there exist $n \times n$ elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A.$$

Proof of Theorem 1 ───────────────────────────────────

The first result is the observation that left multiplication of matrix A_1 by elementary matrix E gives the answer $A_2 = EA_1$ which is obtained by applying the corresponding `combo`, `swap` or `mult` toolkit operation. This fact is discovered by doing examples, then a formal proof can be constructed (not presented here).

Proof of Theorem 2 ───────────────────────────────────

The second result applies the first result multiple times to obtain elementary matrices E_1, E_2, ... which represent the multiply, combination and swap operations performed in the frame sequence which take the First Frame $A_1 = A$ into the Last Frame $A_{k+1} = \mathrm{rref}(A_1)$. Combining the identities

$$A_2 = E_1 A_1, \quad A_3 = E_2 A_2, \quad \ldots, \quad A_{k+1} = E_k A_k$$

gives the matrix multiply equation

$$A_{k+1} = E_k E_{k-1} \cdots E_2 E_1 A_1$$

or equivalently the theorem's result, because $A_{k+1} = \mathrm{rref}(A)$ and $A_1 = A$.

A certain 6-frame toolkit sequence ———————————————————————.

$$A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 6 & 3 \end{pmatrix}$$
         Frame 1, original matrix.

$$A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 3 & 6 & 3 \end{pmatrix}$$
         Frame 2, combo(1,2,-2).

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 3 \end{pmatrix}$$
         Frame 3, mult(2,-1/6).

$$A_4 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{pmatrix}$$
         Frame 4, combo(1,3,-3).

$$A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
         Frame 5, combo(2,3,-6).

$$A_6 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
         Frame 6, combo(2,1,-3). Found $\mathrm{rref}(A_1)$.

Continued

The corresponding 3×3 elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 2, combo(1,2,-2) applied to I.

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 3, mult(2,-1/6) applied to I.

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$ Frame 4, combo(1,3,-3) applied to I.

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix}$$ Frame 5, combo(2,3,-6) applied to I.

$$E_5 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 6, combo(2,1,-3) applied to I.

Toolkit Sequence Details

$A_2 = E_1 A_1$	Frame 2, E_1 equals combo(1,2,-2) on I.
$A_3 = E_2 A_2$	Frame 3, E_2 equals mult(2,-1/6) on I.
$A_4 = E_3 A_3$	Frame 4, E_3 equals combo(1,3,-3) on I.
$A_5 = E_4 A_4$	Frame 5, E_4 equals combo(2,3,-6) on I.
$A_6 = E_5 A_5$	Frame 6, E_5 equals combo(2,1,-3) on I.
$A_6 = E_5 E_4 E_3 E_2 E_1 A_1$	Summary frames 1-6.

Then

$$\mathrm{rref}(A_1) = E_5 E_4 E_3 E_2 E_1 A_1,$$

which is the result of the Theorem.

Fundamental Theorem Illustrated

The summary:

$$A_6 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_1$$

Because $A_6 = \mathrm{rref}(A_1)$, the above equation gives the inverse relationship

$$A_1 = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} \,\mathrm{rref}(A_1).$$

Each inverse matrix is simplified by the rules for constructing E^{-1} from elementary matrix E, the result being

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{rref}(A_1)$$

Theorem 3 (RREF Inverse Method)

$$\operatorname{rref}(< A|I >) =< I|B > \quad \text{if and only if} \quad AB = I.$$

Proof: For *any* matrix E there is the matrix multiply identity

$$E < C|D >=< EC|ED > .$$

This identity is proved by arguing that each side has identical columns. For example, $\operatorname{col}(\text{LHS}, 1) = E \operatorname{col}(C, 1) = \operatorname{col}(\text{RHS}, 1)$.

Assume $C =< A|I >$ satisfies $\operatorname{rref}(C) =< I|B >$. The fundamental theorem of elementary matrices implies $E_k \cdots E_1 C = \operatorname{rref}(C)$. Then

$$\operatorname{rref}(C) =< E_k \cdots E_1 A \mid E_k \cdots E_1 I) =< I|B >$$

implies that $E_k \cdots E_1 A = I$ and $E_k \cdots E_1 I = B$. Together, $BA = I$ and then B is the inverse of A.

Conversely, assume that $AB = I$. Then A has inverse B. The fundamental theorem of elementary matrices implies the identity $E_k \cdots E_1 A = \operatorname{rref}(A) = I$. It follows that $B = E_k \cdots E_1$. Then $\operatorname{rref}(C) = E_k \cdots E_1 < A|I >=< E_k \cdots E_1 A \mid E_k \cdots E_1 I >=< I|B >$.