

## **Rank, Nullity, Dimension and Elimination**

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## Last Frame Test

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In a sequence of combo-swap-mult operations, a frame without a signal equation  $\mathbf{0} = \mathbf{1}$  is called the *Last Frame* provided:

- Each nonzero equation has a lead variable. This means the variable has leading coefficient 1 and it appears nowhere else in the system.
- Equations  $\mathbf{0} = \mathbf{0}$  appear at the end.
- Variables within an equation appear in variable list order.
- Nonzero equations are listed with lead variables in variable list order.

## Last Frame Example

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Let's assume variable list order  $x, y, z$ .

$$\begin{cases} x + y & = 2, \\ & z = 3, \\ & 0 = 0. \end{cases}$$

The lead variables are  $x, z$  and the free variable is  $y$ .

### **Definition 1 (Reduced Echelon System)**

A linear system which passes the last frame test is called a **reduced echelon system**.

### **Definition 2 (Rank and Nullity)**

Assume the last frame test has been passed. Then

- Rank = number of lead variables,
- Nullity = number of free variables (non-lead variables).

### **Determining the rank and nullity of a system** \_\_\_\_\_

- Display a frame sequence whose first frame is that system, and whose last frame passes the *Last Frame Test*. The last frame is a *reduced echelon system*.
- The rank of the system is the number of lead variables in the last frame.
- The nullity of the system is the number of variables minus the rank. It is exactly the free variable count.

## **Theorem 1 (Rank and Nullity)**

The following equations hold:

**lead variable count + free variable count = number of variables,**

or

**rank + nullity = number of variables.**

### **Definition 3 (Dimension)**

The **dimension** of a system of equations is the number of free variables, which is the number of invented symbols assigned in the Last Frame Algorithm.

The case of a homogeneous  $3 \times 3$  system gives geometric meaning to the term **dimension**. In this case, dimension 0 is the unique solution case, geometrically a point, and dimensions 1, 2, 3 enumerate the possibilities for the general solution to describe a line, plane or all of space.

Dimension is devoid of geometric meaning. It is a gross measure of the complexity of the solution set of the system of equations.

## Elimination

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The elimination algorithm applies at each algebraic step one of the three toolkit rules **swap**, **multiply** and **combination**.

- The objective of each algebraic step is to **increase the number of lead variables**. The process stops when a signal equation (typically  $0 = 1$ ) is found. Otherwise, it stops when no more lead variables can be found, and then the last system of equations is a **reduced echelon system**. A detailed explanation of the process appears *infra*.
- Reversibility of the algebraic steps means that no solutions are created nor destroyed throughout the algebraic steps: the original system and all systems in the intermediate steps have *exactly the same solutions*.
- The final reduced echelon system has either a unique solution or infinitely many solutions. In both cases we report the **general solution**. In the infinitely many solution case, the **last frame algorithm** is used to write out a general solution.

## **Theorem 2 (Elimination)**

Every linear system has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated application of the toolkit rules **swap**, **multiply** and **combination**.

## An Elimination Algorithm

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An equation is said to be **processed** if it has a lead variable. Otherwise, the equation is said to be **unprocessed**.

1. If an equation “ $0 = 0$ ” appears, then move it to the end. If a signal equation “ $0 = c$ ” appears ( $c \neq 0$  required), then the system is inconsistent. In this case, the algorithm halts and we report **no solution**.
2. Identify the **first symbol**  $x_r$ , in variable list order  $x_1, \dots, x_n$ , which appears in some unprocessed equation. Apply the **multiply** rule to insure  $x_r$  has leading coefficient one. Apply the **combination** rule to eliminate variable  $x_r$  from all other equations. Then  $x_r$  is a **lead variable**: the number of lead variables has been increased by one.
3. Apply the **swap** rule repeatedly to move this equation past all processed equations, but before the unprocessed equations. Mark the equation as **processed**, e.g., replace  $x_r$  by boxed symbol  $\boxed{x_r}$ .
4. Repeat steps 1–3, until all equations have been processed once. Then lead variables  $x_{i_1}, \dots, x_{i_m}$  have been defined and the last system is a reduced echelon system.

**1 Example (Elimination)** Solve the system.

$$\begin{aligned}w + 2x - y + z &= 1, \\w + 3x - y + 2z &= 0, \\x + z &= -1.\end{aligned}$$

**Solution**

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The answer using the natural variable list order  $w, x, y, z$  is the standard general solution

$$\begin{aligned}w &= 3 + t_1 + t_2, \\x &= -1 - t_2, \\y &= t_1, \\z &= t_2, \quad -\infty < t_1, t_2 < \infty.\end{aligned}$$

Details appear in the next few slides.

## Elimination Details

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We will apply the three rules **swap**, **multiply** and **combination** for equivalent equations to obtain a frame which passes the Last Frame Test. Then we apply the Last Frame Algorithm to obtain the general solution of the system.

Let's mark processed equations with a box enclosing the lead variable ( $w$  is marked  $\boxed{w}$ ).

$$\begin{array}{rcl} \boxed{w} + 2x - y + z & = & 1 \\ \boxed{w} + 3x - y + 2z & = & 0 \\ & x & + z = -1 \end{array} \quad \mathbf{1}$$

$$\begin{array}{rcl} \boxed{w} + 2x - y + z & = & 1 \\ 0 + x + 0 + z & = & -1 \\ & x & + z = -1 \end{array} \quad \mathbf{2}$$

$$\begin{array}{rcl} \boxed{w} + 2x - y + z & = & 1 \\ & x & + z = -1 \\ & & 0 = 0 \end{array} \quad \mathbf{3}$$

$$\begin{array}{rcl} \boxed{w} + 0 - y - z & = & 3 \\ & \boxed{x} & + z = -1 \\ & & 0 = 0 \end{array} \quad \mathbf{4}$$

## Documentation of the steps

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- ❶ Original system. Identify the variable order as  $w, x, y, z$ .
- ❷ Choose  $w$  as a lead variable. Eliminate  $w$  from equation 2 by using `combo(1, 2, -1)`.
- ❸ The  $w$ -equation is processed. Let  $x$  be the next lead variable. Eliminate  $x$  from equation 3 using `combo(2, 3, -1)`.
- ❹ Eliminate  $x$  from equation 1 using `combo(2, 1, -2)`. Mark the  $x$ -equation as processed. **Last Frame Test** passed.

The four frames make the **frame sequence** which takes the original system into the last frame. Basic exposition rules apply:

1. Variables in an equation appear in variable list order.
2. Equations inherit variable list order from the lead variables.

## Last Frame Algorithm

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The last frame of the sequence, which must pass the Last Frame Test, is used to write out the general solution, as follows.

$$\begin{aligned} \mathbf{w} &= 3 + y + z \\ \mathbf{x} &= -1 - z \\ y &= t_1 \\ z &= t_2 \end{aligned}$$

$$\begin{aligned} w &= 3 + t_1 + t_2 \\ x &= -1 - t_2 \\ y &= t_1 \\ z &= t_2 \end{aligned}$$

Solve for the lead variables  $\mathbf{w}$ ,  $\mathbf{x}$ . Assign invented symbols  $t_1$ ,  $t_2$  to the free variables  $y$ ,  $z$ .

Back-substitute free variables into the lead variable equations to display the standard general solution.

**Requirement:** variables must be listed in variable list order  $[w, x, y, z]$ .