Example. Three Possibilities with Symbol $\boldsymbol{k}$

Determine all values of the symbol $\boldsymbol{k}$ such that the system below has one of the *Three Possibilities*

(1) No solution,

(2) Infinitely many solutions,

(3) A unique solution.

Display all solutions found.

$$\begin{array}{rcrcl} x & + & ky & = & 2, \\ (2-k)x & + & y & = & 3. \end{array}$$

The solution of this problem involves construction of three frame sequences, the last frame of each resulting in one classification among the Three Possibilities:

(1) No solution, (2) Unique solution, (3) Infinitely many solutions.

The plan, for each of the three possibilities, is to obtain a triangular system by application of swap, multiply and combination rules. The three possibilities are detected by (1) A signal equation "$\mathbf{0} = \mathbf{1}$," (2) One or more free variables, (3) Zero free variables.

Shared Frames

The three expected frame sequences share these initial frames:

$$\begin{array}{rcrcr} x & + & ky & = & 2, \\ (2-k)x & + & y & = & 3. \end{array}$$

Frame 1.

Original system.

$$\begin{array}{rcrcr} x & + & ky & = & 2, \\ 0 & + & [1+k(k-2)]y & = & 2(k-2)+3. \end{array}$$

Frame 2.

`combo(1,2,k-2)`

$$\begin{array}{rcrcr} x & + & ky & = & 2, \\ 0 & + & (k-1)^2y & = & 2k-1. \end{array}$$

Frame 3.

Simplify.

At this point, we identify the values of k that split off into the three possibilities.

Analysis of the Frames

$$\begin{array}{rcrcr} x & + & ky & = & 2, \\ 0 & + & (k-1)^2 y & = & 2k-1. \end{array}$$

Frame 3.

Simplify.

- There will be a signal equation if the second equation of Frame 3 has no variables, but the resulting equation is not "$0 = 0$." This happens exactly for $k = 1$. The resulting signal equation is "$0 = 1$." We conclude that one of the three frame sequences terminates with the *no solution case*. This frame sequence corresponds to $k = 1$.
- Otherwise, $k \neq 1$. For these values of k, there are zero free variables, which implies a unique solution.
- A by-product of the analysis is that the *infinitely many solutions* case never occurs!

The Conclusion

The three frame sequences reduce to two frame sequences. One sequence gives no solution and the other sequence gives a unique solution.

The Three Answers

(1) There is no solution only for $k = 1$.

(2) Infinitely many solutions never occur for any value of k.

(3) For $k \neq 1$, there is a unique solution

$$x = 2 - k(2k-1)/(k-1)^2,$$
$$y = (2k-1)/(k-1)^2.$$