Systems of Second Order Differential Equations
Cayley-Hamilton-Ziebur

- Characteristic Equation
- Cayley-Hamilton
  - Cayley-Hamilton Theorem
  - An Example
- Euler’s Substitution for $\ddot{\mathbf{u}} = A\mathbf{u}$
- The Cayley-Hamilton-Ziebur Method for $\ddot{\mathbf{u}} = A\mathbf{u}$
**Definition 1 (Characteristic Equation)**

Given a square matrix $A$, the **characteristic equation** of $A$ is the polynomial equation

$$\det(A - \lambda I) = 0.$$ 

The determinant $\det(A - \lambda I)$ is formed by subtracting $\lambda$ from the diagonal of $A$.

The polynomial $p(x) = \det(A - xI)$ is called the **characteristic polynomial** of matrix $A$.

- If $A$ is $2 \times 2$, then $p(x)$ is a quadratic.
- If $A$ is $3 \times 3$, then $p(x)$ is a cubic.
- The determinant is expanded by the cofactor rule, in order to preserve factorizations.
Characteristic Equation Examples

Create \( \det(A - xI) \) by subtracting \( x \) from the diagonal of \( A \).
Evaluate by the cofactor rule.

\[
A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, \quad p(x) = \begin{vmatrix} 2 - x & 3 \\ 0 & 4 - x \end{vmatrix} = (2 - x)(4 - x)
\]

\[
A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}, \quad p(x) = \begin{vmatrix} 2 - x & 3 & 4 \\ 0 & 5 - x & 6 \\ 0 & 0 & 7 - x \end{vmatrix} = (2-x)(5-x)(7-x)
\]
Theorem 1 (Cayley-Hamilton)
A square matrix $A$ satisfies its own characteristic equation.

If $p(x) = (-x)^n + a_{n-1}(-x)^{n-1} + \cdots + a_0$, then the result is the equation

$(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0I = 0$,

where $I$ is the $n \times n$ identity matrix and $0$ is the $n \times n$ zero matrix.

The 2 × 2 Case

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and for $a_1 = \text{trace}(A)$, $a_0 = \det(A)$ we have $p(x) = x^2 + a_1(-x) + a_0$. The Cayley-Hamilton theorem says

$A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. 
Cayley-Hamilton Example

Assume

\[
A = \begin{pmatrix}
  2 & 3 & 4 \\
  0 & 5 & 6 \\
  0 & 0 & 7
\end{pmatrix}
\]

Then

\[
p(x) = \begin{vmatrix}
  2 - x & 3 & 4 \\
  0 & 5 - x & 6 \\
  0 & 0 & 7 - x
\end{vmatrix} = (2 - x)(5 - x)(7 - x)
\]

and the Cayley-Hamilton Theorem says that

\[
(2I - A)(5I - A)(7I - A) = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}.
\]
Euler’s Substitution and the Characteristic Equation

**Definition.** Euler’s Substitution for the second order equation $\vec{u}'' = A\vec{u}$ is

$$\vec{u} = \vec{v}e^{rt}.$$ 

The symbol $r$ is a real or complex constant and symbol $\vec{v}$ is a constant vector.

**Theorem 2 (Euler Solution Equation from Euler’s Substitution)**

Euler’s substitution applied to $\vec{u}'' = A\vec{u}$ leads directly to the equation

$$|A - r^2 I| = 0.$$ 

This is perhaps the premier method for remembering the characteristic equation for the second order vector-matrix equation $\vec{u}'' = A\vec{u}$.

**Proof:** Substitute $\vec{u} = \vec{v}e^{rt}$ into $\vec{u}'' = A\vec{u}$ to obtain $r^2 e^{rt} \vec{v} = A\vec{v}e^{rt}$. Cancel the exponential, then $r^2 \vec{v} = A\vec{v}$. Re-arrange to the homogeneous system $(A - r^2 I) \vec{v} = \vec{0}$. This homogeneous linear algebraic equation has a nonzero solution $\vec{v}$ if and only if the determinant of coefficients vanishes: $|A - r^2 I| = 0$. 
Cayley-Hamilton-Ziebur Method for Second Order Systems

Theorem 3 (Cayley-Hamilton-Ziebur Structure Theorem for $\ddot{\mathbf{u}} = A\dot{\mathbf{u}}$)
The solution $\mathbf{u}(t)$ of second order equation $\ddot{\mathbf{u}}(t) = A\dot{\mathbf{u}}(t)$ is a vector linear combination of Euler solution atoms corresponding to roots of the equation $\det(A - r^2 I) = 0$.

The equation $|A - r^2 I| = 0$ is formed by substitution of $\lambda = r^2$ into the eigenanalysis characteristic equation of $A$.

In symbols, the structure theorem says

$$\mathbf{u} = \mathbf{d}_1 A_1 + \cdots + \mathbf{d}_k A_k,$$

where $A_1, \ldots, A_k$ are Euler solution atoms corresponding to the roots $r$ of the determining equation $|A - r^2 I| = 0$. Therefore, all vectors in the relation have real entries. However, only $2n$ entries of vectors $\mathbf{d}_1, \ldots, \mathbf{d}_k$ are arbitrary constants, the remaining entries being dependent on them.
Proof of the Cayley-Hamilton-Ziebur Theorem

Consider the case when \( A \) is \( 2 \times 2 \) (\( n = 2 \)), because the proof details are similar in higher dimensions. Expand \(|A - xI| = 0\) to find the characteristic equation \( x^2 + cx + d = 0 \), for some constants \( c, d \). The Cayley-Hamilton theorem says that \( A^2 + cA + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). Let \( \vec{u} \) be a solution of \( \vec{u}''(t) = A\vec{u}(t) \). Multiply the Cayley-Hamilton identity by vector \( \vec{u} \) and simplify to obtain

\[
A^2\vec{u} + cA\vec{u} + d\vec{u} = \vec{0}.
\]

Using equation \( \vec{u}''(t) = A\vec{u}(t) \) backwards, we compute \( A^2\vec{u} = A\vec{u}'' = \vec{u}''' \). Replace the terms of the displayed equation to obtain the relation

\[
\vec{u}''' + c\vec{u}'' + d\vec{u} = \vec{0}.
\]

Each component \( y \) of vector \( \vec{u} \) then satisfies the 4th order linear homogeneous equation \( y^{(4)} + cy^{(2)} + dy = 0 \), which has characteristic equation \( r^4 + cr^2 + d = 0 \). This equation is the expansion of determinant equation \(|A - r^2I| = 0\). Therefore \( y \) is a linear combination of the Euler solution atoms found from the roots of this equation. It follows then that \( \vec{u} \) is a vector linear combination of the Euler solution atoms so identified. This completes the proof.